HOMEWORK #2

1) (a) + (b)

Show that $\Delta_\pm = \frac{1}{2} m (\dot{x}^2 \pm \dot{y}^2) + \frac{1}{2} m\omega^2 (x^2 \pm \alpha y^2)$ are conserved. $(\Delta_+ \equiv E,\ \Delta_- \equiv \Delta)$

$$\frac{d}{dt}\Delta_\pm = m\left(\dot{x}\ddot{x} \pm \dot{y}\ddot{y} + \omega^2 x\dot{x} \pm \alpha y\dot{y}\right)$$

EOM: $\ddot{x} + \omega^2 x = 0$

$\ddot{y} + \alpha\omega^2 y = 0$

$$\Rightarrow \frac{d}{dt}\Delta_\pm = m\omega^2\left(-x\dot{x} \mp \alpha y\dot{y} + x\dot{x} \pm \alpha y\dot{y}\right) = 0$$

(c) By def. point transformations do not depend on velocities. Hence $\frac{\partial Q_1}{\partial t}$ has no power of $\dot{x}$ in it (similarly for Q_2 and $\dot{y}$). On the other hand the Lagrangian is quadratic in velocities. $\frac{\partial L}{\partial \dot{x}}$ has one power of $\dot{x}$. As a result T is linear in velocities.

(d) Δ is not linear in $\dot{x}$ and $\dot{y}$ because of the term $\frac{1}{2}m(\dot{x}^2 - \dot{y}^2)$

(e) For $\alpha = 1$ the problem has rotation symmetry (isotropic). Hence we expect angular momentum to be conserved. Indeed

$L_z = x\dot{y} - y\dot{x}$ has the desired form

L_z is linear in velocities and cannot be written as a linear combination of E and Δ which have no linear (in $\dot{x}, \dot{y}$) terms at all.

2) (a) $-\underbrace{\frac{\delta L}{\delta \phi}}_{-c^2\partial_x^2\phi} + \underbrace{\frac{\partial}{\partial t}\frac{\delta L}{\delta(\partial_t \phi)}}_{\partial_t^2\phi} = 0 \quad \Rightarrow (\partial_t^2 - c^2\partial_x^2)\phi = 0$

Here I used the rule for functional differentiation of derivatives

$$\frac{\delta(\partial_\mu \phi)}{\delta \phi} = -\partial_\mu$$

(Of course this is abuse of notation. This has to act on something. ~~Ay~~ You can show this using integration by parts.)

(b) $\partial_t \phi = \partial_{t'}\phi \frac{\partial t'}{\partial t} + \partial_{x'}\phi \frac{\partial x'}{\partial t}$

$\partial_x \phi = \partial_{t'}\phi \frac{\partial t'}{\partial x} + \partial_{x'}\phi \frac{\partial x'}{\partial x}$

Using the Lorenz transformation we have

$\frac{\partial t'}{\partial t} = \gamma$, $\frac{\partial x'}{\partial t} = -\beta\gamma c$, $\frac{\partial t'}{\partial x} = -\frac{\beta\gamma}{c}$, $\frac{\partial x'}{\partial x} = \gamma$

$$(\partial_t \phi)^2 = (\partial_{t'}\phi)^2\gamma^2 - 2\beta\gamma^2 c\,(\partial_{t'}\phi)(\partial_{x'}\phi) + \beta^2\gamma^2c^2(\partial_{x'}\phi)^2$$

$$c^2(\partial_x \phi)^2 = (\partial_{t'}\phi)^2\frac{\beta^2\gamma^2}{\cancel{c^2}} - 2\beta\gamma^2 c\,(\partial_{t'}\phi)(\partial_{x'}\phi) + \gamma^2c^2(\partial_{x'}\phi)^2$$

$$(\partial_t \phi)^2 - c^2(\partial_x\phi)^2 = \underbrace{\gamma^2(1-\beta^2)}_{1}\left[(\partial_{t'}\phi)^2 - c^2(\partial_{x'}\phi)^2\right]$$

(c) $J = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial t} \\ \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial t} \end{vmatrix} = \begin{vmatrix} \gamma & -\beta\gamma c \\ -\frac{\beta\gamma}{c} & \gamma \end{vmatrix} = \gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1$

Jacobian is one. Hence $dx\,dt = dx'dt'$

Together with part (b) this implies that the action is Lorentz invariant

(d) Again $\partial_t = \frac{\partial x'}{\partial t}\partial_{x'} + \frac{\partial t'}{\partial t}\partial_{t'} = -\beta\gamma c\,\partial_{x'} + \gamma\partial_{t'}$

$c\partial_x = c\frac{\partial x'}{\partial x}\partial_{x'} + c\frac{\partial t'}{\partial x}\partial_{t'} = c\gamma\partial_{x'} - \beta\gamma\partial_{t'}$

$$\partial_t^2 - c^2\partial_x^2 = \underbrace{\gamma^2(1-\beta^2)}_{1}\partial_{t'}^2 - c^2\underbrace{\gamma^2(1-\beta^2)}_{1}\partial_{x'}^2 + \underbrace{(2\beta\gamma^2c - 2\beta\gamma^2c)}_{0}\partial_{t'}\partial_{x'} = \partial_{t'}^2 - c^2\partial_{x'}^2$$

JS/3.3) $\vec{r}\,' = R\vec{r}$ where $R = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}$, $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$(\vec{r}\,')^2 = (x')^2 + (y')^2 = (\vec{r}\,')^T(\vec{r}\,') = (R\vec{r})^T(R\vec{r}) = \vec{r}^{\,T} R^T R \vec{r}$$

Similarly $\dot{x}'^2 + \dot{y}'^2 = (\dot{\vec{r}}\,')^2 = \dot{\vec{r}}^{\,T} R^T R \dot{\vec{r}}$

Note that $R^T R = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}\begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

$\Rightarrow (r')^2 = r^2 \Rightarrow r' = r$ also $\dot{x}'^2 + \dot{y}'^2 = \dot{x}^2 + \dot{y}^2$

JS/3.8) (a) Note that rotations preserve the length of vectors. ~~[illegible]~~ Thats why they must have the property $R^T R = \mathbb{1}$ as was demonstrated above. But this also ensures that the kinetic energy term is invariant, which was shown above too $\left(\dot{x}'^2 + \dot{y}'^2 = \dot{\vec{r}}^{\,T} \underbrace{R^T R}_{1} \dot{\vec{r}} = \dot{\vec{r}}^{\,T}\dot{\vec{r}} = \dot{x}^2 + \dot{y}^2 \right)$

<u>Noether Theorem</u>:

$$T = \frac{\partial L}{\partial \dot{q}_\mu}\delta q^\mu - \delta\Phi$$

↳ In this case we don't have this term.

$T \propto \dot{x}y - x\dot{y} \propto L_z$	for rotations around z-axis (hence $\delta z = 0$) ($\delta x = \epsilon y$, $\delta y = -\epsilon x$)
$T \propto \dot{y}z - y\dot{z} \propto L_x$	" " " x-axis
$T \propto \dot{z}x - z\dot{x} \propto L_y$	" " " y-axis

(b) We are told that $\mathcal{L}$ is invariant under rotations. So we can ~~[illegible]~~ apply Noether Theorem immediately. Since $V(r)$ is independent of $\dot{x}$ and $\dot{y}$, the derivation of part (a) applies here.

J.S/3.23) $K = \frac{1}{2} m\left[(\rho\dot{\phi})^2 + \dot{\rho}^2 + \dot{z}^2\right]$ cylindrical coordinates.

$$\mathcal{L} = \frac{1}{2} m\left(\rho^2\dot{\phi}^2 + \dot{\rho}^2 + \dot{z}^2\right) - \frac{1}{2} k\left(\rho^2 + z^2\right) \underbrace{- \lambda_1(\rho - b) - \lambda_2(z - a\phi)}_{\text{constraints.}}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = m\rho^2\ddot{\phi} - \lambda_2 a = 0 \qquad (1)$$

$$\rho\text{-equation} \quad = m\ddot{\rho} - m\dot{\phi}^2\rho + k\rho + \lambda_1 = 0 \qquad (2)$$

$$z\text{-eq.} \quad = m\ddot{z} + kz + \lambda_2 = 0 \qquad (3)$$

$(1) + (\rho = b) \Longrightarrow \quad \ddot{\phi} = \frac{\lambda_2 a}{m b^2}$. or $\lambda_2 = \frac{m b^2}{a}\ddot{\phi}$

$(3) + (z = a\phi) \Longrightarrow \quad ma\ddot{\phi} + ka\phi + \lambda_2 = 0$

$$m\ddot{\phi}\left(1 + \frac{b^2}{a^2}\right) + k\phi = 0$$

$$\ddot{\phi} + \left(\frac{a^2 k/m}{a^2+b^2}\right)\phi = 0$$

EOM of harmonic oscillator with freq. $\omega^2 = \frac{ak}{a^2+b^2}$

$(2) \Longrightarrow \lambda_1 = -kb + m\dot{\phi}^2\rho$

$$\Rightarrow \boxed{\begin{aligned} -\lambda_1 &= +kb - m\dot{\phi}^2\rho \\ -\lambda_2 &= -\frac{m b^2}{a}\ddot{\phi} \end{aligned} \quad \text{where } \phi(t) = \underbrace{A \sin(\omega t + \psi)}_{\text{determined by initial conditions.}}}$$

Forces of constraints have a relative "−" sign due to way we put the constraint in the Lagrangian.