Physics 601 Homework 5---Due Friday October 9

1. Consider a separable system with n degrees of freedom which can be written in terms of action-angle variables. This problem explores the conditions for which the system has periodic motion for all choices of initial conditions.
   a. Show that a sufficient condition for this is that the Hamiltonian can be written in the form $H(J_1,J_2 \ldots J_n) = h(k_1 J_1 + k_2 J_2 + \ldots k_n J_n)$ where h is a function and $k_1,k_2,\ldots,k_n$ are integers.
   b. Is the condition in a. necessary? If so demonstrate this. If not explicit construct a counterexample.

2. Consider the Hamiltonian for a particle in an external electromagnetic field $H = \frac{1}{2}(\vec{p} - q\vec{A})^2 + \Phi(\vec{x})$. One can make a time-independent gauge transformation to a new Hamiltonian $H' = \frac{1}{2}(\vec{p} - q\vec{A}')^2 + \Phi'(\vec{x})$ where $\Phi' = \Phi$ and $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$. This describes the same physical system so one expects that this can equally be thought of as a canonical transformation. Indeed it can, one can rewrite H' as $H' = \frac{1}{2}(\vec{p}' - q\vec{A})^2 + \Phi(\vec{x}')$ with $\vec{x}' = \vec{x}$, $\vec{p}' = \vec{p} - \vec{\nabla}\Lambda$. Show that this transformation really is canonical by demonstrating that canonical Poisson bracket relations are satisfied.

3. Consider a particle of mass, *m*, and charge, *q*, constrained to move in the x-y plane in a magnetic field which is constant in both space and timespace of magnitude B_0 oriented in the positive z direction. There is a gauge with $\Phi = 0, \quad A_x = -\frac{1}{2}B_0 y \quad A_y = \frac{1}{2}B_0 x \quad A_z = 0$.
   a. Starting with the Hamiltonian in Cartesian coordinates, show that the Hamiltonian in polar coordinates is given by
   $$H(r,\theta,p_r,L) = \frac{p_r^2}{2m} + \frac{\left(L - \frac{1}{2}qB_0 r^2\right)^2}{2mr^2}.$$
   b. Now *L* is cyclic and hence conserved. Show that for circular orbits around the origin, *L* is completely determined by the radius of the orbit and determine the relation between them.
   c. Show that for these orbits $L = \frac{q\Phi_m}{2\pi}$ where Φ_m is magnetic flux enclosed by the orbit.

4. Consider the problem discussed in 3. modified however so that the strength of the magnetic field while constant in space is a function of time. To be concrete assume that $B_0 = \begin{cases} B_i & \text{for} \quad t < 0 \\ B_i + \left(B_f - B_i\right)\frac{t}{\tau} & \text{for} \quad 0 \le t < \tau \\ B_f & \text{for} \quad \tau \le t \end{cases}$ that assume the field strength starts at B_i and beginning at *t=0* grows linear for a time τ ending at field strength of B_f with

$B_f > B_i$. Now suppose that for *t<0* the particle makes a circular orbit around the origin with a radius *R*. Suppose further that at *t=0* the particle is on the positive axis (θ=0).

a. Suppose that field is turned on very slowly (adiabatically). In this case one expects on physical grounds that the orbit will remain centered at the origin with its radius adjusting due to the changed magnetic field. What is the magnetic flux enclosed by the final orbit (i.*e.* for $\tau \le t$)? What is its radius of the orbit. Hint: Do NOT solve explicitly solve the equation of motion for r.
b. Explain briefly on physical grounds why one expects adiabatic changes to keep the orbit centered on the origin. State the condition for the validity of the adiabatic approximation which keeps the orbit centered around the origin.

5. A standard result in undergraduate relativity is the velocity addition formula: $v = \frac{v_1 + v_2}{1 + v_1 v_2}$. This typically obtained by taking the product of two Lorentz transformations and the doing some algebra. A more straight forward way to obtain this use the four velocity: $u^\mu = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$ $\gamma = \frac{1}{\sqrt{1 - \vec{v}^2}}$. Suppose I start with a particle moving with velocity $\vec{v}_1$ with an associated four-velocity $u_1^{\,\mu} = \begin{pmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ \gamma_1 v_{1y} \\ \gamma_1 v_{1z} \end{pmatrix}$.
Suppose one boosts to a new frame by running to the left (-*x* direction) with a velocity which corresponds to a Lorentz transformation $\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma_2 & v_2 \gamma_2 & 0 & 0 \\ v_2 \gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
The four-velocity in the new frame is $u^\nu = \Lambda^\mu{}_\nu u_1^{\,\nu}$.
   a. From the transformed four velocity find v_x, v_y, v_z in the new frame.
   b. For the case where $\vec{v}_1$ is entirely along the x direction, show that $v = \frac{v_1 + v_2}{1 + v_1 v_2}$.

6. Consider relativistic transformations restricted to one spatial direction. In that case, the velocity can be specified by a single number from -1 to 1 (in units with *c*=1). It is convenient to introduce the "rapidity" η with the property that $v = \tanh(\eta)$. Note that while *v* is restricted from -1 to 1, η goes from -∞ to ∞.

a. Show that the 4-velocity is given by $u^\mu = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \\ 0 \\ 0 \end{pmatrix}$.

b. Show that while velocities do not add linearly in relativity rapidities do. That is one obtains the relativistic velocity addition formula by taking $\eta = \eta_1 + \eta_2$. In a certain sense this turns out to be just the hyperbolic trig version of the angle addition formula for two-dimensional rotations.