Physics 601 Homework 10---Due Friday November 6

1. Consider a rigid body with the following mass density:
   $\mu(\vec{x}) = \mu_0 \exp\left(-\frac{1}{2}\left(\frac{z^2}{L^2} + \frac{x^2}{L^2} + \frac{y^2}{L^2} + \frac{xy}{L^2}\right)\right)$
   a. Find the moment of inertia tensor.
   b. Find the three principal moments of inertia.

2. Rotation matrices are those dimension square matrices with determinant one that satisfy $\vec{\vec{R}}^T \vec{\vec{R}} = \vec{\vec{1}}$. Show that these form a group.

3. This problem concerns rotations about the z axis.
   a. Show that the rotation about the z axis: $\vec{\vec{R}} = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ can be written as $\vec{\vec{R}} = \exp\left(-\vartheta \vec{\vec{T}}^z\right)$ where $T^z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
   b. Show that $\cos(\vartheta) = \frac{1}{2}\mathrm{tr}(\vec{\vec{R}}) - \frac{1}{2}$

4. There is a general theorem by Euler that any rotation matrix can be represented as a rotation about one given axis. Thus by analogy to problem 3a. it can be written as $\vec{\vec{R}} = \exp\left(-\gamma \hat{n} \cdot \vec{\vec{\vec{T}}}\right) = \exp\left(-\gamma\left(n_x \vec{\vec{T}}_x + n_y \vec{\vec{T}}_y + n_z \vec{\vec{T}}_z\right)\right)$ where $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is the unit vector specify the axis of rotation and γ is the angle specifying the rotation. The purpose of this problem is to find the explicit of the rotation matrix for such a rotation. As a first step note that $\hat{n}$ is completely specified by a polar angle θ and azimuthal angle ϕ. Define $\vec{\vec{R}}_{\hat{n}} \equiv \vec{\vec{R}}_z(\phi)\vec{\vec{R}}_y(\theta)$.
   a. As a first step show that $\hat{n} \cdot \vec{\vec{\vec{T}}} = \vec{\vec{R}}_{\hat{n}} \vec{\vec{T}}^z \vec{\vec{R}}_{\hat{n}}^T$.
   b. Show that $\vec{\vec{R}} = \exp(-\gamma \hat{n} \cdot \vec{\vec{\vec{T}}}) = \vec{\vec{R}}_{\hat{n}} \exp(-\gamma \vec{\vec{T}}^z) \vec{\vec{R}}_{\hat{n}}^T$
   c. Express the nine matrix elements of $\vec{\vec{R}}$ in terms of the polar and azimuthal angles defining $\hat{n}$ and the rotation angle γ.

   Not that this description of the rotation matrices is alternative parameterization to the Euler angles.

5. Suppose we have a rotation specified by the Euler angles $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ (that is $\vec{\vec{R}} = \vec{\vec{R}}_z\left(\frac{\pi}{3}\right)\vec{\vec{R}}_x\left(\frac{\pi}{3}\right)\vec{\vec{R}}_z\left(\frac{\pi}{3}\right)$). Find the angle of rotation about the single axis and the axis $\hat{n}$.