Physics 601 Homework 3---Due Friday September 25

Jose and Slatan problems 5.7, 5.14

In addition:

1. For a particle in 3 dimensions the angular momentum operator is given by $\vec{L} = \vec{x} \times \vec{p}$ where $\vec{x}$ and $\vec{p}$ satisfy are canonical (*i.e.* they satisfy the canonical Poisson bracket relations.
   a. Show that $\{L_i, L_j\} = \varepsilon_{ijk} L_k$ where *i,j,k* take on the values *x,y,z.* (Note that this is isomorphic to the commutators for the angular momentum in quantum mechanics. For those with mathematically inclinations, this is the Lie algebra SO(3).)
   b. Show that $\{\vec{L}, L^2\} = 0$ (that is that $\{L_i, L^2\} = 0$ for all *i*) where $L^2 = \vec{L} \cdot \vec{L}$.
   c. Show that $\{\vec{L}, f(r)\} = 0$ where *f* is an arbitrary function and $r = \sqrt{\vec{x} \cdot \vec{x}}$.

Note that parts b. &c. reflect a deeper result $\{\vec{L}, s\} = 0$ for any scalar *s*. This reflects the fact that the angular momentum is the generator of rotations. (See problem 2)

2. This problem and the subsequent 2 discusses the physics in 5.3.4 in a rather different (and I hope more transparent) language. Label our canonical variables by a phase-space vector

$$\vec{\xi} = \begin{pmatrix} q_1 \\ q_2 \\ \cdots \\ q_n \\ p_1 \\ p_2 \\ \cdots \\ p_{n1} \end{pmatrix}$$ which satisfies canonical Poisson brackets $\left\{\xi^i, \xi^j\right\} = \omega^{ij}$

now suppose that there is a one parameter continuous family of canonical transformations depending on one parameter ε : $\vec{\eta}(\vec{\xi}; \varepsilon)$ which by construction also satisfies the canonical Poisson bracket relations $\left\{\eta^i, \eta^j\right\} = \omega^{ij}$.

   a. Show that $\vec{\eta}(\vec{\xi}; \varepsilon)$ is canonical if it satisfies the conditions $\frac{d\eta^i}{d\varepsilon} = \{\eta^i, g\}$ with $\vec{\eta}(\vec{\xi}; 0) = \vec{\xi}$ for some g. Note that this is the same form as usual Hamiltonian time εϖολυτιον with t replaced by ε and H replaced by g. The function g is called the generator of the transformation.

b. The time evolution of the phase-space position under Hamiltonian gives a transformation from an initial point in phase space to a subsequent one. That is $\vec{\xi}(t)$ is really a function the time and the initial conditions: $\vec{\xi}(\vec{\xi}_0,t)$. Note that the initial conditions are a set of phase space variable which are canonical. Now let me define a canonical transformation a canonical transformation $\vec{\eta}(\vec{\xi},T)$ which has the same functional relation as $\vec{\xi}(\vec{\xi}_0,t)$. That is $\vec{\eta}(\vec{\xi},T)$ corresponds to the value the point in phase space that any point in phase space evolves into from $\vec{\xi}$ a time T later. Using a show that for any T, $\vec{\eta}(\vec{\xi},T)$ is canonical with H acting as the generator of time translations.

3. This problem uses the result in problem 2 in the context of a 6-dimension phase space associated with a single particle in 3 dimensions. The canonical variables are x,y,z,p_x,p_y,p_z . In this problem various transformations with clear physical meanings are proposed. State the physical meaning, show that they are canonical and find the generator of the transformation and show it satisfies the relation in 2.. In all of these ε is a constant.

$X = x + \varepsilon$	$X = x\cos(\varepsilon) + y\sin(\varepsilon)$	$X = x$	$X = (1+\varepsilon)x$
$Y = y$	$Y = -x\sin(\varepsilon) + y\cos(\varepsilon)$	$Y = y$	$Y = (1+\varepsilon)y$
$Z = z$	$Z = z$	$Z = z$	$Z = (1+\varepsilon)z$
$P_x = p_x$	$P_x = p_x\cos(\varepsilon) + p_y\sin(\varepsilon)$	$P_x = p_x + \varepsilon$	$P_x = (1+\varepsilon)^{-1}p_x$
$P_y = p_y$	$P_y = -p_x\sin(\varepsilon) + p_y\cos(\varepsilon)$	$P_y = p_y$	$P_y = (1+\varepsilon)^{-1}p_y$
$P_z = p_z$	$P_z = p_z$	$P_z = p_z$	$P_z = (1+\varepsilon)^{-1}p_z$
$a.$	$b.$	$c.$	d

4. Show that if a one parameter family of canonical transformations leaves the Hamiltonian invariant then its generator is conserved. That is given a canonical transformation $\vec{\eta}(\vec{\xi};\varepsilon)$ with $H(\vec{\eta}(\vec{\xi};\varepsilon)) = H(\vec{\xi})$ then the generator is conserved.