Equation of motion (EOM) for free particle in a rotating coordinate system using Lagrangian formalism

- Original (inertial) frame with (Cartesian) coordinate $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$ ($\hat{x}$... being unit vectors)
- Lagrangian in terms of $\bar{r}$ is

$$L(\bar{r}, \dot{\bar{r}}) = \tfrac{1}{2} m |\dot{\bar{r}}|^2, \text{ where } \dot{\bar{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \quad \dots (1)$$

- New (non-inertial) coordinates: $\bar{r}'$ obtained by rotating $\bar{r}$ about $\underline{z}$-axis by ωt (constant angular speed) so that

$$x' = x\cos\omega t + y\sin\omega t\,;\; y' = y\cos\omega t - x\sin\omega t$$
$$z' = z \quad \dots (2)$$

- Plug inverse of Eq.(2), i.e., (x, y, z) in terms of (x', y', z'), into Eq.(1) in order to find Lagrangian in terms of new coordinates:

$$\tilde{L}(\bar{r}', \dot{\bar{r}}') = \frac{1}{2} m \left[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2\right]$$

$$= \frac{1}{2} m \left|\dot{\bar{r}}' + \bar{\omega} \times \bar{r}'\right|^2, \quad \dots (3)$$

where in 2^{nd} line above, $\bar{\omega} = \omega\hat{z}\ (= \omega\hat{z}')$

- **Checks**: $\boxed{1^{st}}$ line of Eq.(3) can be re-written in terms of (x, y, z) using Eq.(2), i.e.,

$$(\dot{x}' - \omega y') = \dot{x}\cos\omega t - \omega x\sin\omega t + \dot{y}\sin\omega t + \omega y\cancel{\cos\omega t}$$
$$- \omega y\cancel{\cos\omega t} + \omega x\sin\omega t = \dot{x}\cos\omega t + \dot{y}\sin\omega t$$

Similarly, $(\dot{y}' + \omega x') = \dot{y}\cos\omega t - \dot{x}\sin\omega t$ (2)

Thus, we get

$$(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 = \dot{x}^2 + \dot{y}^2$$

and obviously $\dot{z}' = \dot{z}$ so that 1st line of Eq.(3) is indeed same as Eq.(1)

— As far as 2nd line of Eq.(3) goes, we see that

$$\dot{\bar{r}}' + \bar{\omega} \times \bar{r}' = \dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}' + \omega\hat{z}' \times (x'\hat{x}' + y'\hat{y}' + z'\hat{z}')$$

$$= \dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}' + \omega x'\hat{y}' - \omega y'\hat{x}'$$

$$= (\dot{x}' - \omega y')\hat{x}' + (\dot{y}' + \omega x')\hat{y}' + \dot{z}'\hat{z}'$$

so that $|\dot{\bar{r}}' + \bar{\omega} \times \bar{r}'|^2 = (\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2$, i.e., 1st line of Eq.(3)

— Lagrange's equation in new coordinates is

$$d/dt\left(\partial\tilde{L}/\partial\dot{\bar{r}}'\right) = \partial L/\partial\bar{r}' \quad \dots (4)$$

which is really compact/short-hand notation for $\boxed{3}$ separate equations for each of Cartesian coordinates, i.e.,

$$d/dt\left(\partial\tilde{L}/\partial\dot{x}'\right) = \partial\tilde{L}/\partial x' \text{ ; similarly for } y' \,\&\, z' \quad \dots (4)(i)$$

— $\boxed{\text{LHS}}$ of Eq.(4)(i): using $\partial\dot{\bar{r}}/\partial\dot{x} = \hat{x}$ but keeping "tilde" on L (dropping "primes" for simplicity), we have

$$\partial\tilde{L}/\partial\dot{x} = \frac{1}{2}m\frac{\partial}{\partial\dot{x}}\left[(\dot{\bar{r}} + \bar{\omega} \times \bar{r})\cdot(\dot{\bar{r}} + \bar{\omega} \times \bar{r})\right]$$

hits only

$$= \frac{1}{2} m . 2 \quad \hat{x} \cdot (\dot{\bar{r}} + \bar{\omega} \times \bar{r})$$

so that $\frac{d}{dt}\left(\partial \tilde{L}/\partial \dot{x}\right) = m\left(\ddot{\bar{r}} + \bar{\omega} \times \dot{\bar{r}}\right) \cdot \hat{x}$, i.e., $\underbrace{\text{x-component of vector } m(\ddot{\bar{r}} + \bar{\omega} \times \dot{\bar{r}})}$

and similarly for $\boxed{y, z}$ components

i.e., compactly, we can write

$$d/dt \; \partial \tilde{L}/\partial \dot{\bar{r}} = m\left(\ddot{\bar{r}} + \bar{\omega} \times \dot{\bar{r}}\right) \quad \dots (5)$$

– Onto RHS of Eq. (4) (i) : using $\partial \bar{r}/\partial x = \hat{x}$, we get

$$\partial \tilde{L}/\partial x = \tfrac{1}{2} m \, \partial/\partial x \left[(\dot{\bar{r}} + \bar{\omega} \times \bar{r}) \cdot (\dot{\bar{r}} + \bar{\omega} \times \bar{r})\right]$$

hits (only)

$$= \frac{1}{2} m . 2 \underbrace{(\dot{\bar{r}} + \bar{\omega} \times \bar{r})}_{\bar{A}} \cdot (\underbrace{\bar{\omega}}_{\bar{B}} \times \underbrace{\hat{x}}_{\bar{C}})$$

Using the property $\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{C} \cdot (\bar{A} \times \bar{B})$, this gives

$\partial \tilde{L}/\partial x = m \; \hat{x} \cdot \left[(\dot{\bar{r}} + \bar{\omega} \times \bar{r}) \times \bar{\omega}\right]$, i.e x-component of vector $m\left[(\dot{\bar{r}} + \bar{\omega} \times \bar{r}) \times \bar{\omega}\right]$

and similarly for y, z components so that in short-hand notation :

$$\partial \tilde{L}/\partial \bar{r} = m\left[\dot{\bar{r}} + (\bar{\omega} \times \bar{r})\right] \times \bar{\omega} = m\left[\dot{\bar{r}} \times \bar{\omega} + (\bar{\omega} \times \bar{r}) \times \bar{\omega}\right]$$

$$= -m\left[\bar{\omega} \times \dot{\bar{r}} + \bar{\omega} \times (\bar{\omega} \times \bar{r})\right] \quad \dots (6)$$

Equating (5) & (6) [as per Eq. (4)] (and "restoring" primes),

$$m\left[\ddot{\bar{r}}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}') + 2 \, \bar{\omega} \times \dot{\bar{r}}'\right] = 0 \quad \dots (7)$$

centrifugal force coriolis force

[Of course, Eq. (7) can also be obtained directly by plugging Eq. (2) into $m \ddot{\bar{r}} = 0$, i.e., EOM in inertial frame.]