Short $\boxed{\text{note}}$ on Noether's theorem

– Consider a general transformation of coordinates which has an infinitesimal version (called a continuous transformation):

$$\underbrace{q_i'(t)}_{\text{new coordinate}} = \underbrace{q_i(t)}_{\text{old coordinate}} + \overbrace{\alpha}^{\text{small}} \underbrace{\Delta q_i(t)}_{\text{shift in } q_i} \quad \cdots (1)$$

– Here, α is a small parameter and $\Delta q_i(t)$ is a general function, i.e., not just of all coordinates $[q(t)]$, but possibly their time-derivatives (see example later) [cf. change of coordinates studied earlier in the context of showing that it keeps form of Lagrange's equation unchanged.].

– Such a transformation is called a symmetry of the Lagrangian if it leaves the actual/final equations of motion (EOM) – and not just it's "form" – the same, i.e., (laws of) physics is unchanged

– Now, for EOM to remain the same, it is not necessary for Lagrangian (L) [or action (S)] itself to be invariant; rather it suffices for L to transform by total time derivative of some function K (see later for an example), i.e.,

$$\tilde{L} = L + \alpha \frac{dK}{dt} \quad \text{or} \quad L \to L + \alpha \Delta L, \text{ with}$$

(again, this is original L re-written in terms of new coordinates)

$$\Delta L = \frac{dK}{dt} \quad \dots (2)$$

Why? The point is that S then transforms as

$$S = \int_{t_0}^{t_f} dt\, L \longrightarrow S + \alpha \int_{t_0}^{t_f} dt \frac{dK}{dt} = S + \alpha\left[K(t_f) - K(t_0)\right]$$

where the extra term is not relevant for extremizing S (i.e., while varying paths). Thus, actual EOM is not modified.

— Next, we compare above ($\alpha \Delta L$) to what we get by carrying out the transformation of Eq. (1) on L, i.e., (as usual) using chain rule gives

$$\left(\frac{\partial L}{\partial q_i}\right)\left[\alpha \Delta q_i(t)\right] + \left(\frac{\partial L}{\partial \dot{q}_i}\right) \alpha \frac{d[\Delta q_i(t)]}{dt}$$

(an $\boxed{\text{implicit}}$ assumption, i.e., no change/shift of time, $\boxed{t}$, is made here: we will return to it in an example)

$$= \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\left[\alpha \Delta q_i(t)\right] + \left(\frac{\partial L}{\partial \dot{q}_i}\right) \alpha \frac{d[\Delta q_i(t)]}{dt}$$

(where Lagrange's equation was used in 1st term)

$$= \alpha\, d/_{dt}\left[\frac{\partial L}{\partial \dot{q}_i} \Delta q_i(t)\right] \quad \dots \boxed{(3)}$$

(as usual, repeated indices are summed over)

... which we can set to $\alpha \dfrac{dK}{dt}$ [using Eq. (2)]

so that

$$d/_{dt}\left[\frac{\partial L}{\partial \dot{q}_i} \Delta q_i - K\right] = 0, \text{ i.e.,}$$

the quantity

$$c \equiv \left[\frac{\partial L}{\partial \dot{q}_i} \Delta q_i - K\right] \quad \cdots (4)$$

is conserved (or is a constant of motion)

Examples

① Uniform translation of a closed system of particles with potential energy, $V(|\bar{r}_i - \bar{r}_j|)$ that was discussed earlier: in this case, q_i's are $\bar{r}_i$ (i.e., position vectors of particles), with $\bar{r}_i{}' = \bar{r}_i + \alpha \bar{n}$ (where $\bar{n}$ is unit vector in direction of translation), i.e., $\Delta q_i = \bar{n}$"

Clearly, L itself is unchanged here (i.e., homogeneity of space) so that K in Eq. (2) is 0, giving conserved quantity (as from Eq. (4))

$$c = \left(\frac{\partial L}{\partial \dot{\bar{r}}_i} \Delta \bar{r}_i - 0\right) = \underbrace{\left(\frac{1}{2} m_i 2 \dot{\bar{r}}_i\right)}_{\text{from } \frac{1}{2} m \dot{\bar{r}}_i^2 \text{ part of } L} \cdot \bar{n}$$

= component of total (linear) momentum of system along $\bar{n}$ (as expected)

② Similarly, isotropy of space, i.e., uniform rotation of the system keeps L invariant ($K = 0$), with $\bar{r}_i{}' = \bar{r}_i + \alpha(\bar{n} \times \bar{r}_i)$ ($\bar{n}$ being unit vector along axis of rotation)

$\Rightarrow$ from Eq. (4), $c = \frac{\partial L}{\partial \dot{\bar{r}}_i} \cdot (\bar{n} \times \bar{r}_i)$, i.e., component of (total) angular momentum along $\bar{n}$ is conserved

invariance under (simply)

(3) Almost finally, homogeneity of $\boxed{\text{time}}$, i.e., (time translation : t'(new time coordinate) $= t(\text{old}) - \alpha$

so that

$q_i'(t')$, i.e., new coordinate evaluated at new time

$= q_i(t)$ (old ... at old ...)

i.e., $q'(t-\alpha) = q(t)$ or $q_i'(t) = q_i(t+\alpha)$ (note "sign flip" for α)

which for α small becomes

$q_i'(t) \simeq q_i(t) + \alpha \dot{q}_i(t)$, i.e., as per Eq.(1),

we have $\Delta q_i(t) = \dot{q}_i(t)$ (again, $\dot{q}$ and not just q's in Δq)

Unlike in previous 2 examples, in this case L does transform ; in fact similarly to $q_i(t)$ above, i.e.,

$L'(t) = L(t+\alpha)$ so that K [in Eq. (2)] $= L$

$\approx L(t) + \alpha\, dL/dt$

conserved quantity from Eq. (4) is then

$c = \left(\frac{\partial L}{\partial \dot{q}} \underbrace{\dot{q}}_{\Delta q} - L \right)$, i.e., $\boxed{\text{energy-function}}$ defined earlier [= actual energy for system of particles with potential dependent only on position]

Finally (!), recall that $\boxed{\partial L/\partial t = 0}$ was actually used in $\boxed{\text{earlier}}$ proof of $dh/dt = 0$ (see sec. 2.7 of GPS); indeed, we also need this here : (in general there is an extra term in Eq. (3) for ΔL : $\boxed{\alpha\, \partial L/\partial t}$ for time translation (i.e, $t \to t - \alpha$) ... which is set to 0 here.