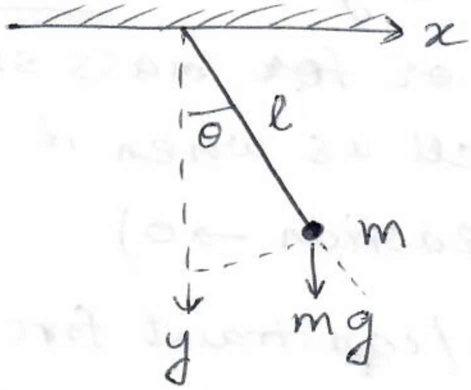
Phys 601 | Dr. Agashe

Note on Lagrange-multiplier method for simple pendulum : solving for constraint force as function of coordinate (angle)



Lagrange multiplier

– Lagrangian with constraint included:

$$L' = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy + \tfrac{1}{2}(x^2+y^2-l^2)\lambda$$

– change to polar coordinates : $x = r\sin\theta$; $y = r\cos\theta$

so that (using $\dot{x} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$; $\dot{y} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$)

$$L' = \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + \tfrac{1}{2}\lambda(r^2 - l^2),$$

giving Lagrange's equations:

for λ : $r = l$ (i.e., λ enforces constraint) $\Rightarrow \dot{r} = 0$, $\ddot{r} = 0$

for r : $\frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} = m\ddot{r} = 0 = \frac{\partial L'}{\partial r} = mr\dot{\theta}^2 + \lambda r + mg\cos\theta$ (set to l)

i.e., $\lambda = -m\dot{\theta}^2 - mg/l\,\cos\theta$ ... (1)

for θ : $d/dt\left(\partial L'/\partial\dot{\theta}\right) = \frac{1}{2} m \underbrace{r^2}_{=l} 2\ddot{\theta} = \partial L'/\partial\theta = -mg\underbrace{r}_{=l}\sin\theta$

i.e., $\ddot{\theta} = -g/l\,\sin\theta$ ...(2)

In principle, we can solve Eq.(2) for $\theta(t)$; then plug this solution into Eq. (1) to get $\lambda(\underline{t})$

- However, sometimes it is enough to obtain $\lambda(\underline{\theta})$, i.e., constraint force as function of coordinate, e.g., in this case, we can then determine at what angle will string become slack (tension $\rightarrow 0$)... or for mass sliding on a surface, this will tell us when it will fall off surface (normal reaction $\rightarrow 0$)

[That $\lambda \propto$ tension/constraint force is clear from equation leading to (1), where $mg\cos\theta$ is component of gravitational force along $\hat{r}$; so is $(-)\lambda l$, which must then be tension...]

- For above purpose, we solve Eq (2) as follows:

$$\frac{d/dt\,(\dot{\theta}^2)}{2\dot{\theta}} = \ddot{\theta} = \frac{d/dt\,(\cos\theta)}{\dot{\theta}}\, g/l \text{, i.e.,} \quad \cdots (3)$$

$\dot{\theta}^2 = 2g/l \cos\theta + c$ (constant), where c is fixed by initial/boundary condition, e.g., $\dot{\theta} = 0$ at maximum value of θ ($= \theta_{max}$) $\Rightarrow c = -2g/l \cos\theta_{max}$

so that Eq. (1) [plugging in Eq. (3)] gives

$$\lambda(\theta) = \underbrace{-2mg/l\cos\theta + 2mg/l\cos\theta_{max}}_{-m\dot{\theta}^2} - mg/l\cos\theta$$

$$\boxed{\lambda(\theta) = -3mg/l\cos\theta + 2mg/l\cos\theta_{max}}$$