Central force motion/Kepler problem

This short note summarizes our discussion in the lectures of various aspects of the motion under central force, in particular, the Kepler problem of inverse square-law (gravitational) force: for more details, refer to the notes that you should have taken during lectures or GPS chapter 3.

1 Reducing 2-body motion to effective 1-body, that too with 2 d.o.f and 1st order differential equations

(a). To begin with, we re-write the coordinates of the 2 bodies in terms of their relative coordinate (denoted by $\mathbf{r}$) and that of the center of mass (COM) ($\mathbf{R}$). We assume that potential is a function only of $\mathbf{r}$ or its time derivatives. Clearly, the COM then has a constant velocity, thus we neglect its motion and focus on that of $\mathbf{r}$, i.e., effectively 1-body (with reduced mass) moving around a *fixed* center of force (chosen to be at the origin).

(b). Furthermore, we assume that potential depends only on r (i.e., *magnitude* of $\mathbf{r}$), in which case the angular momentum (denoted by $\mathbf{L}$) is a constant of motion.

Using the *direction* of angular momentum being a constant of motion we deduce that motion is in a plane, i.e., we reduce from 3 to 2 dimensions (D)/degrees of freedom (d.o.f.), for which we will use polar coordinates (r and θ).

(c). *Magnitude* of angular momentum:

$$l = mr^2\dot{\theta} \tag{1}$$

is also constant.

(d). Kepler's 2nd law (areal velocity of planets, $= \frac{1}{2}r^2\dot{\theta}$ is constant, for any central force) follows from l being constant.

(e). Since force is conservative (Lagrangian is time-independent), it follows that energy (E) also constant: it is given by sum of potential energy (PE) and kinetic energy (KE)

$$E = V(r) \text{ (``original'' PE)} + \frac{1}{2}m\,\dot{r}^2 \text{ (``radial'' KE)} + \frac{1}{2}m\,r^2\dot{\theta}^2 \text{ (angular KE)} \tag{2}$$

$$= V(r) + \frac{1}{2}m\,\dot{r}^2 + \frac{l^2}{2\,m\,r^2} \text{ using Eq. (1)} \tag{3}$$

From Eq. (3) just above, we have

$$\dot{r} = \sqrt{\frac{2}{m}\left[E - V(r) - \frac{l^2}{2\,m\,r^2}\right]} \tag{4}$$

Using Eqs. (3) and (1), i.e., the two constants of motion, we are thus down to *first* order differential equations (DE's) for r and θ (cf. 2nd order to begin with, i.e., either directly using Newton's laws or via Largrange's equations) (f). In particular, formal solution for $t(r)$ – integrating Eq. (4) – is

$$t = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left[E - V(r') - \frac{l^2}{2\,m\,r'^{\,2}}\right]}} \tag{5}$$

where r_0 is the value of r at $t = 0$.

2 Detour: equivalent 1D potential

(a). In fact, Eq. (3) suggests an equivalent 1D potential, i.e., for motion of r *only*, given by

$$V'(r) \equiv V(r) + \frac{l^2}{2\,m\,r^2} \tag{6}$$

(with kinetic energy of this 1D motion being $m\dot{r}^2/2$ to make up total energy).

(b). The advantage of V' is that we can readily (i.e., without actually solving EOM) deduce qualitative features such as motion is bounded (unbounded) for $E < (>) \, 0$ for a *class* of potentials of the form $V = -k/r^p$, with $k > 0$ and $0 < p < 2$ (which includes inverse-square law force).

(c). In addition, energy and radius of circular orbit for inverse square law ($p = 1$) is given by minimum of $V'(r)$:

$$\begin{aligned} E_{\text{circular}} &= -\frac{m\,k^2}{2\,l^2} \\ r_{\text{circular}} &= \frac{l^2}{m\,k} \end{aligned} \tag{7}$$

3 "Eliminating" time: orbit equation

(a). It is easier to compute (and useful to know) $r(\theta)$ by "canceling" dt between Eqs. (1) and (4) and integrating:

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\,m\,E}{l^2} - \frac{2\,m\,V(r)}{l^2} - \frac{1}{r^2}}} + \text{constant} \tag{8}$$

(b). For Kepler problem, we get conic sections for comets/planets (Kepler's 1st law, valid only for inverse-square law force):

$$\frac{1}{r} = \frac{m\,k}{l^2}\left[1 + e\cos\left(\theta - \theta'\right)\right] \tag{9}$$

with eccentricity given by

$$e = \sqrt{1 + \frac{2\,E\,l^2}{m\,k^2}} \tag{10}$$

and θ' corresponding to closest approach to focus (perihelion). Thus, we have hyperbolic ($e > 1$), parabolic ($e = 1$), elliptical ($0 < e < 1$) and circular ($e = 0$) orbits respectively for $E > 0$, $E = 0$, $E < 0$ and $E = -mk^2/\left(2\, l^2\right)$. All these conclusions match those obtained simply using equivalent 1D potential (i.e., with*out* actually solving EOM, cf. approach just above); in particular, the last result doing so qua*nt*itatively, i.e., being same as 1st line of Eq. (7).

4 More "playing around"

(a). A useful relation can be obtained between the semi-major axis of the ellipse, a (i.e., sum of distances from focus to points of closet and farthest approaches or turning points) and E [based on the quadratic equation for turning points obtained by setting $\dot{r} = 0$ in Eq. (3)]:

$$a \;=\; -\frac{k}{2\,E} \tag{11}$$

In turn, Eqs. (10) and (11) give

$$e \;=\; \sqrt{1 - \frac{l^2}{m\,k\,a}} \tag{12}$$

(b). Kepler's 3rd law (relating time period of orbit of planets, τ to size) can be obtained by equating area from 2nd law to that of ellipse [in terms of a and e from Eq. (12)]:

$$\tau \;=\; 2\,\pi\,a^{\frac{3}{2}}\sqrt{\frac{m}{k}} \tag{13}$$

For $m_{\text{planet}} \ll m_{\text{Sun}}$, we get reduced mass, $m \approx m_{\text{planet}}$. Also, we have k (constant in potential) $=$ $G_N m_{\text{planet}} m_{Sun}$ so that indeed

$$\tau \;\propto a^{\frac{3}{2}} \tag{14}$$

i.e., proportionality constant is *in*dependent of planet.

5 Complete solution: $r(t)$ and $\theta(t)$

5.1 Parabola

In this case, the integral for $t(\theta)$ is easier as follows. In general, plugging $r(\theta)$ from Eq. (9) into Eq. (1) (and rearranging/integrating), we have

$$t \;=\; \frac{l^3}{m\,k^2}\int^{\theta}\frac{d\tilde{\theta}}{\left[1 + e\cos\left(\tilde{\theta} - \theta'\right)\right]^2} + \text{constant} \tag{15}$$

For parabola, i.e., $e = 1$, this simply gives (choosing perihelion to be at $t = 0$ and $\theta = 0$, i.e., setting $\theta' = 0$):

$$t \;=\; \frac{l^3}{2\,m\,k^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right) + \text{constant} \tag{16}$$

which can be inverted to give $\theta(t)$ and plugging this into Eq. (9) can give $r(t)$.

5.2 Ellipse

(a). For this case, we define an intermediate/auxiliary variable, ψ (called "eccentric anomaly", θ being true anomaly) by

$$r \equiv a\,(1 - e \cos\psi) \tag{17}$$

(b). We can get further insight into ψ by comparing $r(\psi)$ in Eq. (17) to $r(\theta)$ in Eq. (9) re-written using Eq. (12) (and assuming $\theta' = 0$), i.e.,

$$r = \frac{a\,(1 - e^2)}{1 + e \cos\theta} \tag{18}$$

This gives

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2} \tag{19}$$

Clearly, as θ goes through a complete revolution/cycle (0 to 2π), so does ψ (justifying it as an "angle"), although (in general) at a *different* (instantaneous) rate than does θ.

(c). The point of introducing a 2nd angle (ψ) is that evaluating integral in $t(r)$ of Eq. (5) is easier using $r(\psi)$, i.e.,

$$\begin{aligned} t &= \sqrt{\frac{m}{2\,k}} \int^{r} \frac{r' dr'}{\sqrt{r' - \frac{r'^{\,2}}{2\,a} - \frac{a}{2}\,(1 - e^2)}} + \text{constant} \\ &= \sqrt{\frac{ma^3}{k}} \int^{\psi} (1 - e\cos\psi')\, d\psi' + \text{constant} \end{aligned} \tag{20}$$

where 1st line is obtained plugging $V(r) = -k/r$ and Eqs. (12) and (11) into Eq. (5) and 2nd line using Eq. (17).

Even though it is not really needed, we can introduce (angular) frequency of oscillation:

$$\omega \equiv \frac{2\,\pi}{\tau} \tag{21}$$

corresponding to a *3rd* (!) angle, called mean anomaly

$$\phi \equiv \omega t \tag{22}$$

which (obviously) goes from 0 to $2\,\pi$ over one period, that too *uniformly* so [cf. θ and ψ are *not* (in general) linear in time.]

We can easily evaluate the integral in 2nd line of Eq. (20) (asssuming $\psi = 0$ at $t = 0$) and use above notation to re-write it as

$$\phi(t)(= \omega t) = \psi - e\sin\psi \tag{23}$$

which is the Kepler equation: solving this transcendental equation gives $\psi(t)$, which when plugged into Eqs. (19) and (17) finally gives us $r(t)$ and $\theta(t)$.

6 Another constant of motion: Laplace-Runge-Lenz vector

(a). Clearly, direction of angular momentum, i.e., $\hat{\mathbf{L}}$, fixes which plane the orbit is in, whereas l and E determine size and shape of orbit, e.g., via Eqs. (11) and (10) for ellipse.

(b). What remains is the (fixed) orientation of major axis of ellipse in plane: it is given by direction of another constant of motion (only for inverse square-law force) called Laplace-Runge-Lenz vector (denoted by $\mathbf{A}$):

$$\mathbf{A} \;=\; p \times \mathbf{L} - m\, k \frac{\mathbf{r}}{r} \tag{24}$$

We can show using EOM of $\mathbf{r}$ that $d\mathbf{A}/dt = 0$.

Also, $\mathbf{A}$ being constant can be used to (re-)derive that orbit is a conic section: this analysis shows that $\mathbf{A}$ points in direction from focus to perihelion (point of closest approach), thus fulfilling its role as above.

(c). $|\mathbf{A}|$ is also constant and is related to E and l

$$|\mathbf{A}| \;=\; m^2 k^2 \left(1 + \frac{2\, E\, l^2}{m\, k^2}\right) \tag{25}$$