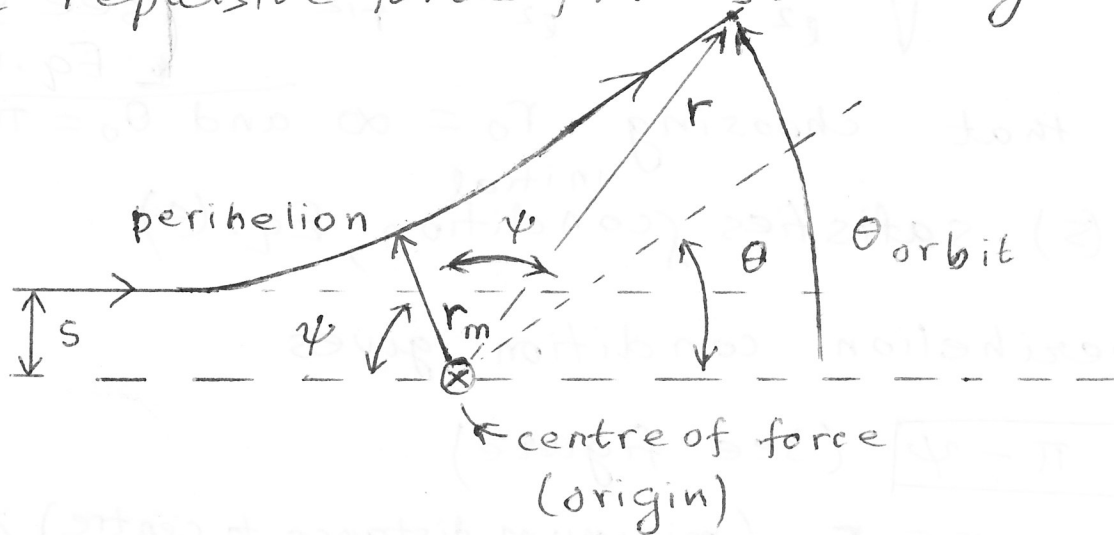


Relating impact parameter(s) to scattering

angle ( $\theta$ ) for equivalent 1-body scattering by central force field (based on GPS sec. 3.10)

For repulsive force, the scattering looks like



At perihelion, angle between position vector and initial direction of particle is  $\psi$  so that (see figure)

$$\theta = \pi - 2\psi \quad \dots (1)$$

and distance from centre of force is  $r_m$ , which (as per equivalent 1-D potential analysis) is given by

$$E = V'(r_m) = V(r_m) + \frac{l^2}{2mr_m^2} \quad \dots (2)$$

$\hookrightarrow$  constant

with  $l$  being "traded" for  $s$  using

$$l = m v_0 s \quad \text{and} \quad E = \frac{1}{2} m v_0^2 \Rightarrow l = s \sqrt{2mE} \quad \dots (3)$$

$\hookrightarrow$  initial / final speed

constant

Also, choose convention for  $\theta_{orbit}$  as shown, i.e.,

( $r = \infty$ )  
initial position corresponds to  $\theta = \pi$  ... (4) (2)

Since the general expression for  $\theta_{\text{orbit}}(r)$  is

$$\theta_{\text{orbit}} = \int_{r_0}^r \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \theta_0 \dots (5)$$

[see GPS, Eq. (3.36)]

we see that choosing  $r_0 = \infty$  and  $\theta_0 = \pi$  in Eq. (5) satisfies initial condition, Eq. (4)

Then, perihelion condition gives

$$\theta_{\text{orbit}} = \pi - \psi \quad (\text{see figure})$$

$$= \int_{\infty}^{r=r_m \text{ (minimum distance to centre)}} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \pi \quad \left[ \begin{array}{l} \text{use} \\ \text{Eq. (4)} \end{array} \right]$$

i.e.,  $\psi = \int_{r_m}^{\infty} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}}$ , giving

[from Eq. (1) and using last of Eq. (3)]

$$\theta(s) = \pi - 2 \int_{r_m}^{\infty} \frac{dr(s)}{r \sqrt{r^2 \left[ 1 - \frac{V(r)}{E} \right] - s^2}} \dots (6)$$

(scattering angle)

with  $r_m$  obtained in terms of  $E$  and  $s$  by solving Eq. (2), plugging in last of Eq. (3)

- A check: from figure, it is clear that final position corresponds to  $\theta_{\text{orbit}} = \theta$ , with  $r \rightarrow \infty$  (like initial position). But, then plugging  $r \rightarrow \infty$  into Eq. (5) - naively - gives  $\theta_{\text{orbit}} = \pi$  (since we chose  $r_0 = \infty$  and  $\theta_0 = \pi$  to satisfy initial condition), i.e., (obviously) same as initial position!

- Resolution: we need to back-up from Eq. (5):  
 rewrite is as (schematically)  $\theta_{\text{orbit}} = \int^r dr' \frac{l / (m r'^2)}{\sqrt{\frac{2}{m} \left[ E - V(r') - \frac{l^2}{2m r'^2} \right]}} \frac{d\theta}{dt} \left\{ \begin{array}{l} \leftarrow \text{drop "orbit" for see previous} \\ \leftarrow \text{ease notes} \end{array} \right.$   
 $= \int dr' / \left( \frac{dr'}{d\theta} \right)$  again (which is how we got Eq. (5) in first place)

- Now,  $r$  and  $\theta_{\text{orbit}}$  are both decreasing as we go from initial to perihelion position (see figure) so that  $dr/d\theta > 0$  along this part of trajectory. Thus, taking positive square root in Eq. (5) was indeed (a posteriori!) justified: recall that, at that stage we only used Eq. (5) in this regime.

- However, from perihelion to final position (i.e.,  $r = r_m$  to  $r = \infty$ ),  $dr/d\theta < 0$  (i.e.,  $r$  is increasing, while  $\theta_{\text{orbit}}$  continues to reduce) so that we should take negative sign for square root, i.e., integral

in Eq. (5) from  $r_m$  to  $\infty$  (again, perihelion (2.5) (ii) to final position) is actually same (not opposite as naively thought to!) that from  $\infty$  to  $r_m$  (along initial to perihelion position). Thus, we get

$$\theta_{\text{orbit for final position}} = 2 \times \int_{\infty}^{r_m} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \pi$$

$$= 2(-\psi) + \pi$$

again, see perihelion position discussion

$$= \theta \text{ from Eq. (1)}$$

again

as expected from figure.

same as integral appearing in perihelion condition used before

If analytic formula for  $r(\theta_{\text{orbit}})$  is known, (3)  
 then  $\theta$  and  $s$  can be related easily using it.

For example, Rutherford scattering, i.e., charged particles ( $z'e$ ) repelled by Coulomb field of nucleus ( $z|e|$ ), so that

force =  $zz'e^2/r^2$ , i.e., inverse square law (like in Kepler problem), but repulsive (vs. attractive for Kepler problem)  $\Rightarrow$  results for Kepler problem can be suitably adapted here, with  $k = -zz'e^2$ , i.e.,  $< 0$  (vs.  $> 0$  in Kepler problem) ... (7)

Orbit equation :

$$\frac{1}{r} = \frac{mk}{l^2} \left[ 1 + \underbrace{\sqrt{1 + \frac{2El^2}{mk^2}}}_{\varepsilon \text{ (eccentricity)}} \cos(\theta_{\text{orbit}} - \theta') \right]$$

becomes  $\frac{1}{r} = \frac{mzz'e^2}{l^2} \left[ -1 - \varepsilon \cos(\theta_{\text{orbit}} - \theta') \right]$  ... (8)

$> 0$   $(\varepsilon > 1)$

with orbit being hyperbola, since  $E > 0$  here

Using same convention for  $\theta_{\text{orbit}}$  as in earlier figure, we see that perihelion (smallest  $r$ ) occurs when [i.e., RHS of Eq 8 is maximized]

$$\cos(\theta_{\text{orbit}} - \theta') = -1 \Rightarrow \underbrace{\pi - \psi - \theta'}_{\text{see figure}} = \cos \pi \Rightarrow \psi = -\theta' \dots (9)$$

asymptotic

Initial position corresponds to  $r \rightarrow \infty$ ,  $\theta_{\text{orbit}} = \pi$   
 so that Eq. (8) gives

$$-1 - \cos(\pi - \theta') \epsilon = 0 \quad \text{or} \quad \cos \theta' = 1/\epsilon (< 1) \dots (10)$$

Eqs. (9) & (10) give

$$\cos \psi = 1/\epsilon \quad \text{or} \quad \left[ \text{using Eq. (11) and algebra} \right]$$

$$\boxed{\cot \theta/2 = \sqrt{\epsilon^2 - 1}} = \boxed{\frac{2Es}{2Z'e^2}} \quad \left[ \text{using last of Eq. (3) in} \right]$$

formula for  $\epsilon$

Some quick checks

(i)  $\boxed{S \rightarrow \infty}$  (fixed  $E$ ) should give very small deflection, since particle doesn't really feel central force, i.e.,  $\theta \rightarrow 0$  ... in agreement with Eq. (11)

Similarly,  $\boxed{S = 0}$ , i.e., particle headed straight toward centre of force will be eventually turned around (no matter what its  $E$  is), since  $V \rightarrow \infty$  as  $r \rightarrow 0$  (particle approaches centre)

... so that  $\theta = \pi$ , again seen in Eq. (11) (and using figure) asymptotic

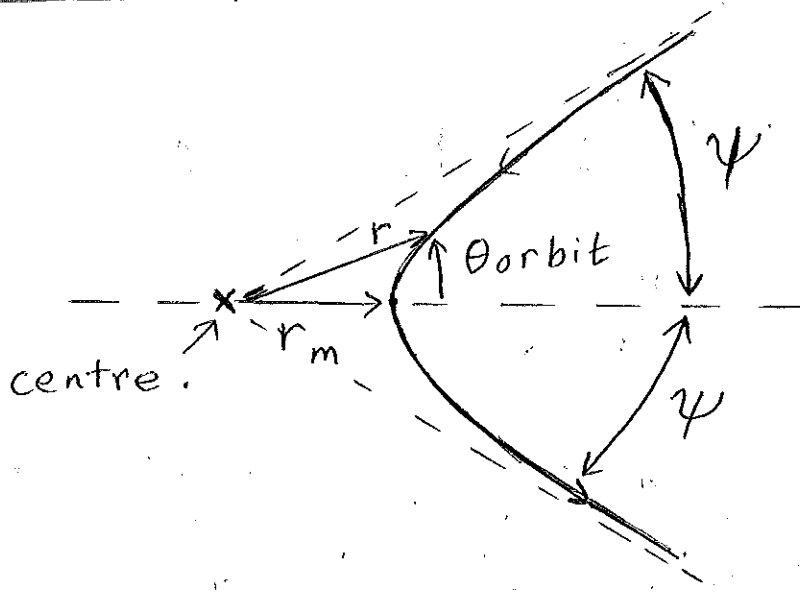
(ii) In Eq. (8),  $\boxed{\theta_{\text{orbit}} = \theta}$  (i.e., final position of particle) should also give  $r \rightarrow 0$ : indeed

plugging  $\theta = \pi - 2\psi$  [as in Eq. (11)], along with  $\theta' = -\psi$  [as in Eq. (9)] in Eq. (8), we find

$$\left( \frac{1}{r} \right) \propto [-1 - \epsilon \cos(\pi - \psi)] = (-1 + \epsilon \cos \psi) \rightarrow 0 \quad \left( \text{using } \cos \psi = 1/\epsilon \right)$$

Clearly, we have  $\boxed{(\pi - 2\psi) \leq \theta_{\text{orbit}} \leq \pi}$ , which ensures  $\boxed{1/r \geq 0}$  from Eq. (8)!

Alternately, as suggested in GPS,



we can "rotate" earlier figure to make it look like above, i.e., perhaps a more standard one, where perihelion is  $\theta_{orbit} = 0$  (cf. figure on 1st page). Then, Eq. (8) gives

$$\text{(maximizing } 1/r) \cos(\theta_{orbit} - \theta') = -1$$

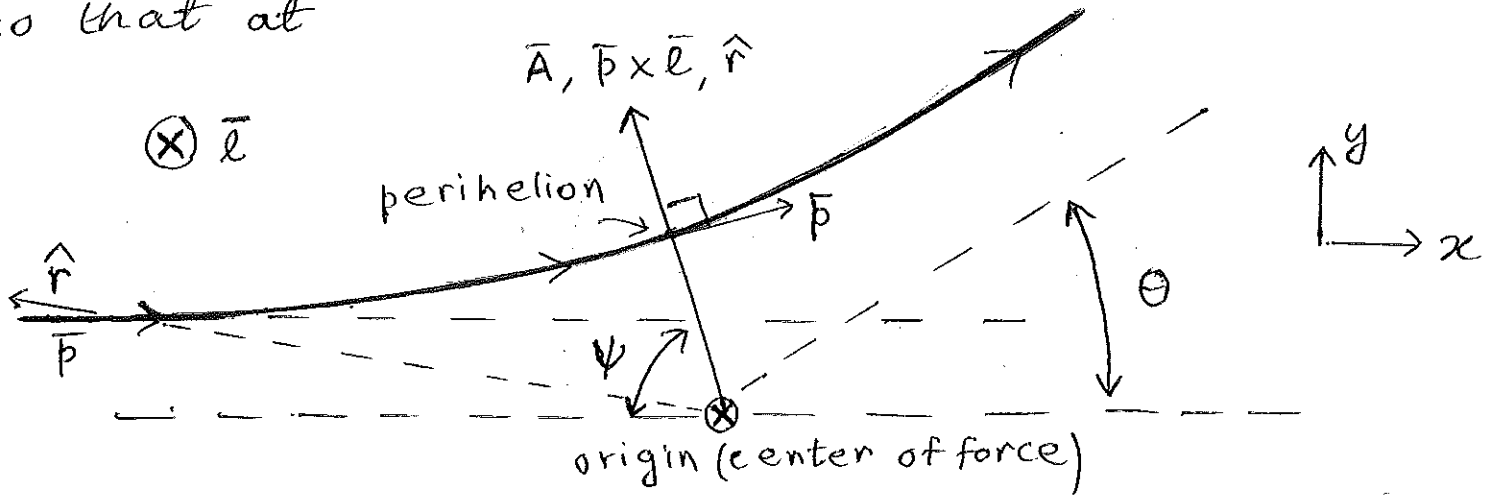
[as in 1st of Eq. (9), but now with  $\theta_{orbit} = 0$  at perihelion by convention]  $\Rightarrow$  we must choose  $\theta' = \pi$

On the other hand initial <sup>(or final)</sup> position of particle ( $r \rightarrow \infty$ ) corresponds to  $\theta_{orbit} = \pm \psi$ . Plugging this in Eq. (8) [with  $\theta' = \pi$  as above] gives

$$\cos \psi = 1/\epsilon \text{ as before...}$$

- A 3<sup>rd</sup> (!) derivation using Laplace-Runge-Lenz vector ( $\bar{A}$ ) is actually the simplest: recall  $\bar{A}$  (= constant of motion) =  $\bar{p} \times \bar{l} - m k \hat{r}$ , with  $\bar{e} = \bar{r} \times \bar{p}$

We have  $\vec{l}$  going into plane of figure below so that at (6)



(a) initial position (asymptotically), we have

$$\begin{aligned} \vec{A} &= (m v_0 l)(\hat{y}) - (m k)(-\hat{x}) \quad (\text{with } x, y \text{-axes as shown}) \\ \text{initial speed} &= m v_0 l \hat{y} - m |k| \hat{x} \quad (\text{since } k < 0 \text{ here}) \end{aligned}$$

(b) while at perihelion  $\vec{p} \times \vec{l}$  is along  $\hat{r}$ , thus so is  $\vec{A}$ , i.e., at an angle  $(\pi - \psi)$  relative to  $x$ -axis

— Equating directions of (a) & (b) [again  $\vec{A}$  is constant of motion], we get

$$-\frac{m|k|}{m v_0 l} = \frac{\cos(\pi - \psi)}{\sin(\pi - \psi)} = -(\tan \psi) \quad \text{--- (1)}$$

Using  $E = \frac{1}{2} m v_0^2$  and  $\epsilon^2 = 1 + \frac{2 E l^2}{(m k^2)}$ , (1)

we then obtain  $(1 + \tan^2 \psi) = \frac{1}{\cos^2 \psi} = 1 + \frac{k^2}{l^2 (2E/m)} = \epsilon^2$

i.e.,  $\cos \psi = 1/\epsilon$  as before

— Again,  $\vec{A}$  points in direction of position vector of perihelion/for this hyperbolic trajectory (just like for ellipse, hyperbola discussed in context of Kepler problem)



at/perihelion/  $\theta_{orbit} = (\pi + \theta')$  (here whether) (7)  
 However, we use convention of Eq. (9), i.e.,  $\theta' = -\psi$ , so that  
 $\theta_{orbit}$  at perihelion =  $(\pi - \psi)$  as in figure on page

(1) [02] we choose figure on page 5, i.e.,  
 $\theta' = \pi$ , so that  $\theta_{orbit}$  at perihelion =  $2\pi = 0$   
 as in that figure] all (i.e. whether or/hyperbola)  
 orbits for planets/comets, cf. for elliptical/closed  
 perihelion was  $\theta_{orbit} = \theta'$  (Kepler problem)

The reason for above difference is flip of sign of  $k$ : again, for Rutherford/repulsive case,

$$\frac{1}{r} = (\text{positive}) [-1 - \epsilon \cos(\theta_{orbit} - \theta')]$$

$$-mk/l^2 = \frac{m z z' e^2}{l^2}$$

so that  $1/r \geq 0$  requires  $\cos(\theta_{orbit} - \theta') < -1/\epsilon$ ;  
 in particular  $1/r$  is maximized (perihelion) for  
 $\cos(\theta_{orbit} - \theta') = -1$ , i.e.,  $\theta_{orbit} = \pi + \theta'$

The above subtlety is also seen while comparing orbit equation derived using  $\bar{A} = \text{constant}$  with that by <sup>actually</sup> solving EOM, i.e.,

$$\frac{1}{r} = \frac{mk}{l^2} \left( 1 + \frac{A}{mk} \cos \tilde{\theta} \right) \rightarrow \text{angle between } \vec{r} \text{ \& } \bar{A}$$

$$= \underbrace{\left[ \frac{-mk}{l^2} \right]}_{>0} \left[ -1 + \underbrace{\left( \frac{\theta A}{mk} \right)}_{>0} \cos \tilde{\theta} \right] \text{ [vs.]}$$

$$\frac{\theta mk}{l^2} [-1 - \epsilon \cos(\theta_{orbit} - \theta')]$$

$$= -\frac{mk}{l^2} \left\{ -1 \oplus \epsilon \cos[\theta_{orbit} - (\pi + \theta')] \right\}$$

so that matching gives

(8)

$$\frac{-A}{mk} (> 0 \text{ because } k < 0 \text{ now}) = \underline{\epsilon} \quad \text{and}$$

$$\tilde{\theta} = \theta_{\text{orbit}} - (\pi + \theta') \text{ or } \boxed{\bar{A} \text{ is along } (\pi + \theta')},$$

again, position vector of perihelion