Solving coupled system (small) oscillations

— Start with Eq. (2.124) of David Tong's (DT) notes, i.e., coupled EOM for small displacements around equilibrium:

$$\ddot{\eta} = F\eta \quad \ldots (1)$$

where η is column vector for perturbations of n coordinates and F is $(n \times n)$ matrix (obtained by suitable Taylor expansion of forces on the n particles: see DT's notes).

— Based on the example of double pendulum (which was worked out in detail in lecture: see also sec. 2.6.1 of DT's notes), we expect that eigenvalues/vectors of matrix F will enable us to solve Eq. (1)

— So, we first solve this eigen-problem:

$$F \mu_a = \lambda_a^2 \mu_a \quad (a = 1 \ldots n) \ldots (2)$$

i.e., determine the eigenvalues λ_a^2 and the corresponding eigenvectors, $\boxed{\mu_a}$ [each of which is a (n-row) column vector]: we can show (see discussion below Eq. 2.128 of DT's notes) that $\boxed{\lambda_a^2}$ are real.

— Construct $n \times n$ matrix (P) out of these eigenvectors: $\boxed{P} \equiv (\mu_1 \; \mu_2 \cdots \mu_n) \ldots (3)$

— It is straightforward to show that ②

$$P^{-1} F P = \lambda^2 \,(\underline{\text{diagonal}}\ \text{matrix}) \quad \cdots (4)$$

where $P^{-1} P = \mathbb{1}$

[Explicitly, let the <u>rows</u> of P^{-1} be denoted by $\boxed{\xi_a}$ $(a = 1 \ldots n)$, i.e., each ξ_a has n columns.

$$P^{-1} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}. \text{ clearly } \underset{\text{row}}{\xi_a} \cdot \underset{\text{column}}{\mu_b} = \delta_{ab} \text{ from } P^{-1}P = \mathbb{1}$$

we have FP [using Eqs.(2) & (3)] $= (\lambda_1^2 \mu_1 \cdots \lambda_n^2 \mu_n)$

so that

$$P^{-1} F P = \begin{pmatrix} \lambda_1^2 \underbrace{\xi_1 \cdot \mu_1}_{=1} & \lambda_2^2 \underbrace{\xi_1 \cdot \mu_2}_{0} & \cdots \\ \lambda_1^2 \underbrace{\xi_2 \cdot \mu_1}_{=0} & \lambda_2^2 \xi_2 \cdot \mu_2 & \\ & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^2 & 0 & \\ 0 & \lambda_2^2 & \cdots \\ & \cdots & \end{pmatrix}$$

as in Eq.(4).]

— Next, we will $\boxed{\text{use}}$ above <u>eigenvalues/vectors</u> to <u>solve</u> Eq.(1). Begin by multiplying (matrix notation) it by P^{-1} from the left:

$$\underbrace{P^{-1}}_{\substack{\text{constant}\\ \text{in time}}} \ddot{\eta} = \underbrace{(P^{-1} F P)}_{\text{use Eq.(4)}} \underbrace{P^{-1}}_{\mathbb{1}} \eta, \text{ i.e., } (\ddot{P^{-1}\eta}) = \underset{\substack{\text{diagonal } n \times n \\ \text{matrix!}}}{\lambda^2} (P^{-1}\eta) \quad \cdots (5)$$

— Define $\boxed{\eta'}$ (column vector with n rows) $\equiv P^{-1}\eta$

so that $\ddot{\eta}'_{i} = \lambda_a^2 \big|_{a=i} \eta'_i$ from Eq.(5), i.e.,

solution is $\eta'_i(t) = A_a e^{\lambda_a t} + B_a e^{-\lambda_a t} \big|_{a=i} \quad \cdots (6)$

— Going <u>back</u> to $\boxed{\eta}$, we have

$$\eta(t) = P\,\eta'(t) = \sum_{a=1\ldots n} \mu_a \left(A_a e^{\lambda_a t} + B_a e^{\lambda_a t}\right) \qquad \ldots(7)$$

given by Eq.(3)

use Eq.(6)

(n-row) column vector

i.e., general solution to Eq.(1) is a superposition of n normal modes as in RHS of Eq.(7) [again, λ_a^2 is eigenvalue corresponding to eigenvector μ_a of F, as in Eq.(2)]