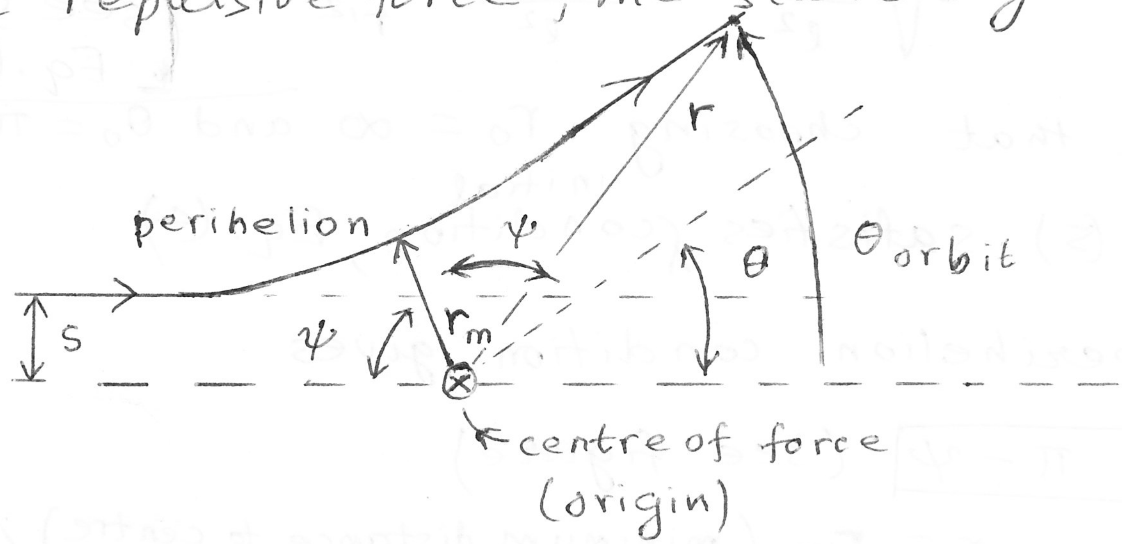
Phys 601 (Fall, 2015) by Dr. Agashe

Relating impact parameter(s) to scattering angle (θ) for equivalent 1-body scattering by central force field (based on GPS sec. 3·10)

For repulsive force, the scattering looks like



At perihelion, angle between position vector and initial direction of particle is $\boxed{\psi}$ so that (see figure)

$$\theta = \pi - 2\psi \quad \cdots (1)$$

and distance from centre of force is r_m, which (as per equivalent 1-D potential analysis) is given by

$$\underbrace{E}_{\text{constant}} = V'(r_m) = V(r_m) + \frac{l^2}{2 m r_m^2} \quad \cdots (2)$$

with l being "traded" for s using

$$\underbrace{l}_{\text{constant}} = m \underbrace{v_0}_{\text{initial / final speed}} s \quad \text{and} \quad E = \tfrac{1}{2} m v_0^2 \Rightarrow l = s\sqrt{2mE} \quad \cdots (3)$$

Also, choose convention for θ orbit as shown, i.e., initial position $(r=\infty)$ corresponds to $\theta = \pi$ ... (4)

Since the general expression for $\theta_{orbit}(r)$ is

$$\theta_{orbit} = \int_{r_0}^{r} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \theta_0 \quad \cdots (5)$$

[see GPS, Eq. (3.36)]

we see that choosing $r_0 = \infty$ and $\theta_0 = \pi$ in Eq. (5) satisfies initial condition, Eq. (4)

Then, perihelion condition gives

$$\boxed{\theta_{orbit} = \pi - \psi} \text{ (see figure)}$$

$$= \int_{\infty}^{r = r_m \text{ (minimum distance to centre)}} \frac{dr'/r'^2}{\sqrt{2mE/\ell^2 - 2mV(r')/\ell^2 - 1/r'^2}} + \pi \quad \text{[use Eq. (4)]}$$

θ_0

i.e., $$\psi = \int_{r_m}^{\infty} \frac{dr'/r'^2}{\sqrt{2mE/\ell^2 - 2mV(r')/\ell^2 - 1/r'^2}}, \text{ giving}$$

[from Eq. (1) and using last of Eq. (3)]

$$\boxed{\theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{dr\,(s)}{r\sqrt{r^2\left[1 - \frac{V(r)}{E}\right] - s^2}}} \quad \cdots (6)$$

(scattering angle)

with $\boxed{r_m}$ obtained in terms of E and s by solving Eq. (2), plugging in last of Eq. (3)

If analytic formula for $\boxed{r(\theta_{orbit})}$ is known, then θ and s can be related easily using it.

For example, Rutherford scattering, i.e., charged particles ($Z'|e|$) repelled by Coulomb field of nucleus ($Z|e|$), so that

force $= ZZ'e^2/r^2$, i.e., inverse square law (like in Kepler problem), but repulsive (vs. attractive for Kepler problem) $\Rightarrow$ results for Kepler problem can be suitably adapted here, with

$\boxed{k = -ZZ'e^2}$, i.e., < 0 (vs. > 0 in Kepler problem) ... (7)

Orbit equation:

$$1/r = \frac{mk}{l^2}\left[1 + \underbrace{\sqrt{1 + \frac{2El^2}{mk^2}}}_{\varepsilon \text{ (eccentricity)}} \cos(\theta_{orbit} - \theta')\right]$$

becomes
$$1/r = \underbrace{\frac{mZZ'e^2}{l^2}}_{>0}\left[-1 - \varepsilon\cos(\theta_{orbit} - \theta')\right] \quad \ldots (8)$$

$(\varepsilon > 1)$

with orbit being hyperbola, since $E > 0$ here.

Using $\boxed{\text{same}}$ convention for θ_{orbit} as in earlier figure, we see that perihelion (smallest r) occurs when [i.e., RHS of Eq. 8 is maximized]

$$\cos(\theta_{orbit} - \theta') = -1 \Rightarrow \underbrace{\pi - \psi}_{\text{see figure}} - \theta' = \cos\pi$$

or $\psi = -\theta'$ ... (9)

Initial asymptotic position corresponds to $r \to \infty$, $\theta_{orbit} = \pi$ so that Eq. (8) gives

$$-1 - \cos(\pi - \theta')\,\varepsilon = 0 \quad \text{or} \quad \cos\theta' = 1/\varepsilon\,(<1) \ldots (10)$$

Eqs. (9) & (10) give

$$\cos\psi = 1/\varepsilon \quad \text{or [using Eq. (1) and some algebra]}$$

$$\boxed{\cot\theta/2 =} \sqrt{\epsilon^2 - 1} = \boxed{\frac{2Es}{ZZ'e^2}} \ldots (11)$$

[using last of Eq (3) in formula for ϵ]

[Some quick checks]

(i) $\boxed{s \to \infty}$ (fixed E) should give very small deflection, since particle doesn't really feel central force, i.e., $\theta \to 0$ ... in agreement with Eq. (11)

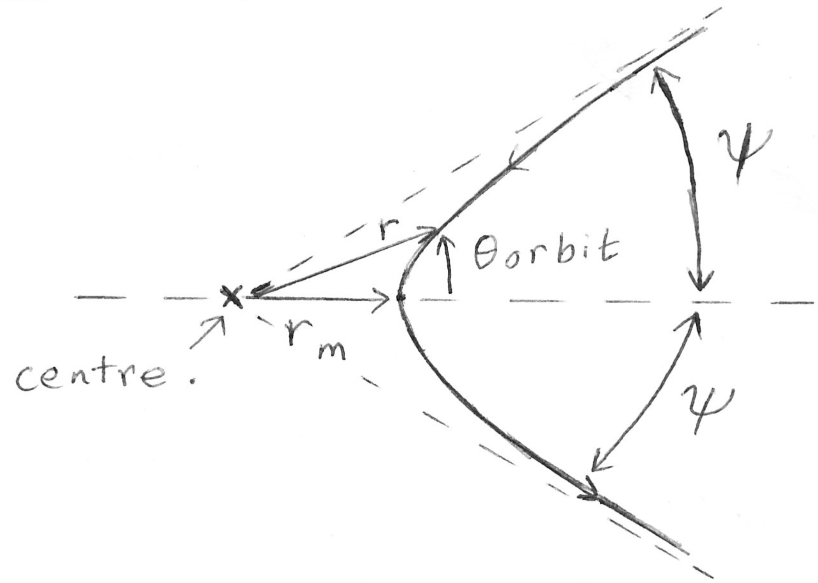
Similarly, $\boxed{s = 0}$, i.e., particle headed straight toward centre of force will be eventually turned around (no matter what its E is), since $V \to \infty$ as $r \to 0$ (particle approaches centre) ... so that $\theta = \pi$ (and using figure), again seen in Eq. (11)

(ii) In Eq. (8), $\boxed{\theta_{orbit} = \theta}$ (i.e., final asymptotic position of particle) should also give $r \to 0$: indeed plugging $\theta = \pi - 2\psi$ [as in Eq. (1)], along with $\theta' = -\psi$ [as in Eq (9)] in Eq. (8), we find

$$1/r \propto [-1 - \varepsilon\cos(\pi - \psi)] = (-1 + \varepsilon\cos\psi) \to 0 \quad \left(\text{using } \cos\psi = 1/\varepsilon\right)$$

Clearly, we have $\boxed{(\pi - 2\psi) \leq \theta_{orbit} \leq \pi}$, which ensures $\boxed{1/r \geq 0}$ from Eq. (8)!

$\boxed{\text{Alternately}}$, as suggested in GPS,



we can "rotate" earlier figure to make it look like above, i.e., perhaps a more standard one, where <u>perihelion</u> is $\boxed{\theta_{orbit} = 0}$ (cf. figure on 1st page). Then, Eq. (8) gives (maximizing $1/r$) $\cos(\theta_{orbit} - \theta') = -1$ [as in 1st of Eq. (9), but now with $\theta_{orbit} = 0$ by convention] $\Rightarrow$ we must choose $\boxed{\theta' = \pi}$

On the other hand <u>initial</u> (or final) position of particle ($r \to \infty$) corresponds to $\boxed{\theta_{orbit} = \pm\psi}$. Plugging this in Eq. (8) [with $\theta' = \pi$ as above] gives

$\cos\psi = 1/\varepsilon$ as before...