Here is the derivation of Eq. 3.7 of David Tong's note using Eqs. 3.5 and 3.6. Plug a suitable version of Eq. 3.6, namely,

$$\omega_b = \frac{1}{2}\epsilon_{bde}\omega_{de} \tag{1}$$

into the factor in the middle term of Eq. 3.7 in order to get

$$\begin{aligned} -\epsilon_{abc}\omega_b &= -\frac{1}{2}\epsilon_{abc}\epsilon_{bde}\omega_{de} \\ &= -\frac{1}{2}\epsilon_{bca}\epsilon_{bde}\omega_{de} \end{aligned} \tag{2}$$

where in 2nd step, the invariance of $\epsilon$ (Levi-Civita symbol) under cyclic permutation is used. Next, the following identity involving $\epsilon$:

$$\epsilon_{bca}\epsilon_{bde} = \delta_{cd}\delta_{ae} - \delta_{ce}\delta_{ad} \tag{3}$$

substituted above gives

$$\begin{aligned} -\epsilon_{abc}\omega_b &= -\frac{1}{2}\left(\omega_{ca} - \omega_{ac}\right) \\ &= \omega_{ac} \end{aligned} \tag{4}$$

where anti-symmetry of $\omega$ is used in 2nd step. This is precisely the factor on RHS of Eq. 3.5.