Phys 601, Fall 2015 (Dr. Agashe)

Note on solving harmonic oscillator problem [i.e., obtaining q, p as functions of t and initial conditions] using Hamilton-Jacobi formalism as discussed in main part of section 4.7 of David Tong (DT)'s lecture notes (cf. section 10.2 of GPS)

— As usual, we have the Hamiltonian (H) given by

$$H = p^2/(2m) + (m\omega^2 q^2)/2, \text{ where } \omega = \sqrt{\frac{k}{m}}, k$$

being spring constant

— Since H is conserved, we can use Hamilton's characteristic function (as denoted in GPS), i.e., time-independent $\boxed{W^0}$ (in DT's notation: Eq. 4.189) which satisfies Eq. 4.190 of DT, i.e.,

$$\boxed{H(q, \partial W^0/\partial q) = E}$$ (constant energy) (thought of as 1st constant of integration)

which for harmonic oscillator becomes

$$(\partial W^0/\partial q)^2/(2m) + (m\omega^2 q^2)/2 = E$$

i.e., $$\partial W^0/\partial q = \sqrt{2mE}\sqrt{1 - \frac{m\omega^2 q^2}{2E}} \quad \ldots (1)$$

[We will find that we do not need to integrate (1), i.e., we will only need $\partial W^0/\partial q$ below.]

— Hamilton's 1st equation (Eq. 4.186 of DT) becomes

$$\boxed{\dot{q} = \partial H/\partial p \Big|_{p = \partial W^0/\partial q}} = \frac{p}{2m}\Big|_{p = \partial W^0/\partial q \text{ of } (1)}$$

$$= \frac{\sqrt{2mE}}{m}\sqrt{1-\frac{m\omega^2 q^2}{2E}} \quad \ldots (2)$$

(again, $\boxed{1^{st}}$ order in t differential equation for $\underline{q}$ $\underline{\text{only}}$)

Integrating (2) gives

$$\int_0^t dt' = \int_{q_0}^{q} \frac{dq'}{\sqrt{1-\frac{m\omega^2 q'^2}{2E}}}\,\frac{m}{\sqrt{2mE}} \quad : \boxed{\text{substitute}}\ \ q' = \sqrt{\frac{2E}{m\omega^2}}\sin\theta$$

q_0: 2nd integration constant (q at $t=0$)

$$= \frac{m}{\sqrt{2mE}} \int_{\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}}q_0\right)}^{\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}}q\right)} \frac{\overbrace{\sqrt{\frac{2E}{m\omega^2}}\cos\theta\, d\theta}^{dq'}}{\cos\theta}$$

$$= \frac{1}{\omega}\left[\sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2E}}\right) - \sin^{-1}\left(q_0\sqrt{\frac{m\omega^2}{2E}}\right)\right]$$

or $$t\omega + \sin^{-1}\left(q_0\sqrt{\frac{m\omega^2}{2E}}\right) = \sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2E}}\right)$$

giving $$\boxed{q(t)} = \sqrt{\frac{2E}{m\omega^2}}\sin\left[\omega t + \sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}}q_0\right)\right] \quad \ldots (3)$$

Plugging (3) in (1) and using $p = \partial W/\partial q$, we get

$$\boxed{p(t)} = \sqrt{2mE}\sqrt{1-\left(\frac{m\omega^2}{2E}\right)\left(\frac{2E}{m\omega^2}\right)\sin^2\left[\omega t + \sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}}q_0\right)\right]}$$

$$= \sqrt{2mE}\cos\left[\omega t + \sin^{-1}\left(q_0\sqrt{\frac{m\omega^2}{2E}}\right)\right] \quad \ldots (4)$$

Eqs.(3) & (4) give evolution of q, p (in terms of energy & initial q) [Equivalently, E can be computed in terms of p_0 & q_0: $E = p_0^2/(2m) + (m\omega^2 q_0^2)/2$]