Phys 420 Exam 2 Solutions

Part I: Wave particle duality

1. If a wavefunction is to represent a particle, what conditions must it satisfy in order for it to be physically meaningful? (*5 points*)

   - It must be continuous.
   - Its derivatives must be continuous.
   - It must be single valued.
   - It must be normalized so that the probability of finding a single particle over all space is 1.

2. A particle of mass m is confined to a one dimensional potential well of width L. The potential inside the well is zero and the potential outside of the well is infinite. Using simple arguments based on DeBroglie waves, show that the energy of the particle can only have discrete values and determine these values. (*25 points*)

   The de Broglie wavelength is given by

   $$\lambda = \frac{h}{p} \tag{1}$$

   The proability of finding the particle outside of the box is zero, therefore, the wavefunction must be zero at the walls of the box in order for the wavefunctions to be continuous. Thus, the box must contain an integer number of half wavelengths. Since the box has a length of L, this condition can be quantified as:

   $$n\left(\frac{\lambda}{2}\right) = L \Rightarrow \lambda = \frac{2L}{n} \tag{2}$$

   Plugging the DeBroglie wavelength into this equation and solving for the momentum:

   $$p = \frac{nh}{2L} \tag{3}$$

   Since there is no potential inside of the well, all of the particle's energy is kinetic. The kinetic energy of a particle of mass m is:

   $$E = \frac{p^2}{2m} = \frac{(nh/2L)^2}{2m} = \frac{n^2h^2}{8mL^2} \tag{4}$$

   And there you have it...we derived the energy quanta simply from the deBroglie wavelength and the boundary conditions.

Part II: Quantum mechanics

Consider two potential wells of width L. One has infinitely high walls as in Part I, the other has finite walls of height U.

1. For a particle with $E < U$, would you expect the allowed wavefunctions for stationary states to be different if it were confined to the finite versus the infinite well? If so, why? (*10 points*)

   Yes, the allowed wavefunctions will differ in two ways:

   - For a finite well, there is a nonzero probability of finding a particle outside of the well. This can be explained either using the uncertianty principle, or by invoking the Schrodinger equation.
   - The stationary states of the finite well will have lower energies than the corresponding states for the infinite well. This is because a larger amount of energy will be needed to confine the particle.

2. If the lowest state for the infinite well is described by $\psi = A \sin kx$ where A is a constant amplitude, write an expression for A. (*10 points*)

   The normalization condition is:

   $$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 \tag{5}$$

   Since the probability of finding the particle outside of an infinite well is zero, this becomes:

   $$A^2 \int_0^L \sin^2 kx \, dx = 1 \tag{6}$$

3. Write an expression for the average position of a particle in this state. (*10 points*)

   Generally:

   $$< x >= \int_{-\infty}^{\infty} \psi(x) \, x \, \psi^*(x) dx \tag{7}$$

   For this example:

   $$< x >= A^2 \int_0^L x \, \sin^2 kx \, dx \tag{8}$$

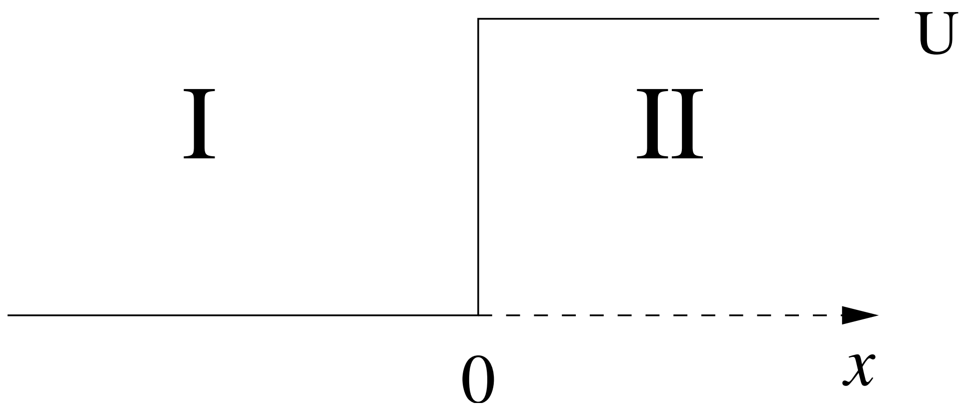

Figure 1: Potential for Part III.

Part III: Potential barriers

Consider the potential barrier with height U shown in Figure 1. A particle with energy $E < U$ approaches the barrier from the left.

1. Write the time independent Schrodinger equation for each of the regions I ($x < 0$) and II ($x > 0$). (*5 points*)

   In the region $x < 0$ there is no potential, so $U = 0$.

   $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x) \tag{9}$$

   In this region the particle is free.

   In the region $x < 0$ there is a potential, so the Schrodinger equation becomes:

   $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi(x) = E\psi(x) \tag{10}$$

2. What is the form of the wavefunction in Region I? in Region II? How would your answer differ if $E > U$? (*5 points*)

   In Region I, assume you have incident and reflected portions of the wave, which are each moving in opposite directions. Neglecting the time evolution:

   $$\psi_I(x) = Ae^{ikx} + Be^{-ikx} \tag{11}$$

   In Region II, you have an evanescent wave which is dying off:

   $$\psi_{II}(x) = Ce^{-\alpha x} \tag{12}$$

IF E were greater than U, the wavefunction would still be sinusoidal in the barrier region, however the kinetic energy and thus the momentum must change since part of its energy will be potential energy. Since $p = \hbar k$, that means that the particle's wavenumber will be different in the barrier region. I will call the wavenumber k' to indicate this:

$$\psi = Ce^{ik'x} \tag{13}$$

You may still get a reflected wave if certian resonant conditions are not met. We derived those resonant conditions in class and did a demonstration to show that you would even get a resonance for a classical wave.

3. What boundary conditions must the wavefunction satisfy at $x = 0$? (*10 points*)

   The wavefunction and its first derivative must be continuous:

$$\psi_I(0) = \psi_{II}(0) \tag{14}$$

$$A + B = C \tag{15}$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0} \tag{16}$$

$$ikA - ikB = -\alpha C \tag{17}$$

4. Use Part 3 to find the reflection coefficient. (*10 points*)

   The reflection coefficient is defined as:

$$R = \frac{(\Psi^*\Psi)_{\text{reflected}}}{(\Psi^*\Psi)_{\text{incident}}} = \frac{B^*B}{A^*A} = \frac{|B|^2}{|A|^2} \tag{18}$$

Use the equations found from the boundary conditions above. From Equation 15 $C = A + B$. Substitute this into Equation 17 and you get:

$$ikA - ikB = -\alpha\,(A + B) \tag{19}$$

Solving for B/A, you get:

$$\frac{B}{A} = \frac{ik + \alpha}{ik - \alpha} \tag{20}$$

Thus the reflection coefficent is:

$$\frac{B^*B}{A^*A} = \frac{(ik+\alpha)}{(ik-\alpha)}\frac{(-ik+\alpha)}{(-ik-\alpha)} = \frac{\alpha^2+k^2}{\alpha^2+k^2} = 1 \tag{21}$$

5. Use the Schrodinger equation to write the wavenumbers in the wavefunctions in Part 2 in terms of the energy. (*10 points*)

   If you take the derivative of the wavefunctions and substitute them into the appropriate time independent Schrodinger equation that you found in 1.) you can solve for k and α.

   Substituting ψ_I and $\frac{d^2\psi_I}{dx^2}$ into the Schrodinger equation with $U = 0$ and solving for k:

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{22}$$

   Substituting ψ_{II} and $\frac{d^2\psi_{II}}{dx^2}$ into the Schrodinger Equation in the presence of a potential and solving for α, you get:

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar} \tag{23}$$