Problem 9.29

From Prob. 9.11, $\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*)$. Here (Eq. 9.176) $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$, $\tilde{\mathbf{B}}^* = \tilde{\mathbf{B}}_0^* e^{-i(kz-\omega t)}$, and, for the TE_{mn} mode (Eqs. 9.180 and 9.186)

$$
\begin{aligned}
B_x^* &= \frac{-ik}{(\omega/c)^2 - k^2}\left(\frac{-m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
B_y^* &= \frac{-ik}{(\omega/c)^2 - k^2}\left(\frac{-n\pi}{b}\right) B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
B_z^* &= B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_x &= \frac{i\omega}{(\omega/c)^2 - k^2}\left(\frac{-n\pi}{b}\right) B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
E_y &= \frac{-i\omega}{(\omega/c)^2 - k^2}\left(\frac{-m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_z &= 0.
\end{aligned}
$$

So

$$
\begin{aligned}
\langle\mathbf{S}\rangle &= \frac{1}{2\mu_0}\left\{\frac{i\pi\omega B_0^2}{(\omega/c)^2 - k^2}\left(\frac{m}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\hat{\mathbf{x}}\right.\\
&\quad + \frac{i\pi\omega B_0^2}{(\omega/c)^2 - k^2}\left(\frac{n}{b}\right)\cos^2\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{y}}\\
&\quad \left. + \frac{\omega k\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{\mathbf{z}}\right\}.\\
\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} &= \boxed{\frac{1}{8\mu_0}\frac{\omega k\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}ab\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right].}
\end{aligned}
$$

along z

[In the last step I used $\int_0^a \sin^2(m\pi x/a)\,dx = \int_0^a \cos^2(m\pi x/a)\,dx = a/2$; $\int_0^b \sin^2(n\pi y/b)\,dy = \int_0^b \cos^2(n\pi y/b)\,dy = b/2$.]

Similarly,

$$
\begin{aligned}
\langle u\rangle &= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)\\
&= \frac{\epsilon_0}{4}\frac{\omega^2\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\
&\quad + \frac{1}{4\mu_0}\left\{B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right.\\
&\quad \left. + \frac{k^2\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right\}.
\end{aligned}
$$

$$
\int\langle u\rangle\,da = \boxed{\frac{ab}{4}\left\{\frac{\epsilon_0}{4}\frac{\omega^2\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \frac{B_0^2}{4\mu_0} + \frac{1}{4\mu_0}\frac{k^2\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right]\right\}.}
$$

These results can be simplified, using Eq. 9.190 to write $\left[(\omega/c)^2 - k^2\right] = (\omega_{mn}/c)^2$, $\epsilon_0\mu_0 = 1/c^2$ to eliminate ϵ_0, and Eq. 9.188 to write $\left[(m/a)^2 + (n/b)^2\right] = (\omega_{mn}/\pi c)^2$:

$$\int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\omega k a b c^2}{8\mu_0 \omega_{mn}^2} B_0^2; \quad \int \langle u \rangle \, da = \frac{\omega^2 a b}{8\mu_0 \omega_{mn}^2} B_0^2.$$

Evidently

$$\frac{\text{energy per unit time}}{\text{energy per unit length}} = \frac{\int \langle \mathbf{S} \rangle \cdot d\mathbf{a}}{\int \langle u \rangle \, da} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} = v_g \text{ (Eq. 9.192).}$$