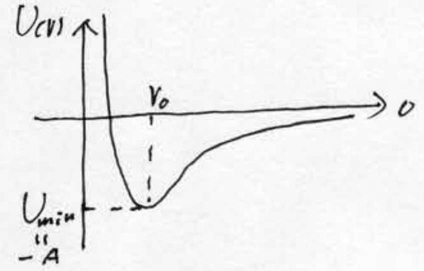
5.2 $U(r) = A[(e^{(R-r)/S} - 1)^2 - 1] = A[e^{2(R-r)/S} - 2e^{(R-r)/S}]$

It's easy to find when $r_0 = R$, $U(r)$ has its minimum $U_{min} = -A$.

And when $r < R$, $U(r)$ decreases with r; while ~~when~~ $r > R$, $U(r)$ increases with r.



For $r \to \infty$, $U(R) \to 0$.

$r = r_0 + x \Rightarrow U(x) = A[(e^{-x/S} - 1)^2 - 1]$

when $x \ll 1$, $U(x) \doteq U(0) + \frac{1}{2} U''(0) \cdot x^2 = -A + \frac{1}{2} \cdot \frac{2A}{S^2} x^2$

$\Rightarrow K = \frac{2A}{S^2}$

5.11. The General form of SHM is:

$$x = A\cos(\omega t - \delta)$$

$$\Rightarrow \begin{cases} x_1 = A\cos(\omega t_1 - \delta) \\ v_1 = -\omega A \sin(\omega t_1 - \delta) \end{cases} \quad \begin{cases} x_2 = A\cos(\omega t_2 - \delta) \\ v_2 = -A\sin(\omega t_2 - \delta) \end{cases}$$

$\Rightarrow A^2 = x_1^2 + \frac{v_1^2}{\omega^2} = x_2^2 + \frac{v_2^2}{\omega^2}$ (or you can also get this by Energy conservation)

$$\Rightarrow \begin{cases} \omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \\ A = \sqrt{\frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}} \end{cases}$$

5.21. $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

Input $x = te^{-\beta t}$ into the equation, we could find that, when $\beta = \omega_0$, it is the solution of the above equation.

In fact, it's the general property of a Linear operator. Let's denote $\mathcal{L}(x) = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$, When $\beta = w_0$, we have $e^{-\beta t}$ as the degenerate solutions:

$$\mathcal{L}(e^{-\beta t}|_{\beta=w_0}) = C\cdot(x - e^{-\beta t}|_{\beta=w_0})^2 = 0$$

$$\Rightarrow \frac{d}{d\beta}\mathcal{L}(e^{-\beta t}|_{\beta=w_0}) = 2C\cdot(x - e^{-\beta t}|_{\beta=w_0})\cdot(-e^{-\beta t}|_{\beta=w_0})' = 0$$

$$\Rightarrow \mathcal{L}\left(\frac{d}{d\beta}(e^{-\beta t})|_{\beta=w_0}\right) = 0$$

$\Rightarrow t e^{-w_0 t}$ is also the solution.

5.33 $\quad x(t) = A\cos(wt-\delta) + e^{-\beta t}[B_1\cos w_1 t + B_2 \sin w_1 t]$

$$\Rightarrow \begin{cases} x_0 = x(0) = A\cos\delta + B_1 \\ v_0 = \dot{x}(0) = Aw\sin\delta - \beta B_1 + w_1 B_2 \end{cases}$$

$$\Rightarrow \begin{cases} B_1 = x_0 - A\cos\beta \\ B_2 = \frac{1}{w_1}[v_0 - Aw\sin\delta + \beta B_1] \end{cases}$$

5.40 $\quad A$ has Maximum

$\Rightarrow A^2$ has Maximum

$$A^2 = \frac{f_0^2}{(w_0^2-w^2)^2+4\beta^2w^2}$$

$$\Rightarrow (w_0^2-w^2)^2+4\beta^2w^2 = [\cancel{w^2+2\beta^2}]w^4 + (4\beta^2-2w_0^2)w^2 + w_0^4 = [w^2-(w_0^2-2\beta^2)]^2 + w_0^4 - (w_0$$

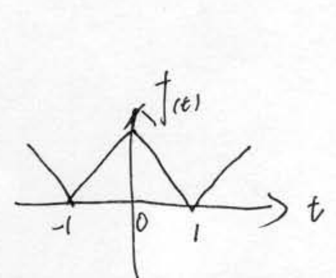
has minimum

Because β is not big, $w_0^2 - 2\beta^2 > 0$, So we could easily find that the required condition is equal to:

$$w^2 = w_0^2 - 2\beta^2$$

$$\Rightarrow w = \sqrt{w_0^2 - 2\beta^2}$$

5.49



clearly, $f(t) = (1-|t|)f_{max}$, for $t \in [-1,1]$

and $f(t+2) = f(t)$

$f(t)$ is even $\Rightarrow$ $f(t)$ is expanded only by $\cos\omega t$

$\Rightarrow b_n = 0.$

$$\Rightarrow \begin{cases} a_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt = \int_0^1 f(t)\,dt = f_{max}\int_0^1 (1-t)\,dt = \frac{f_{max}}{2} \\ a_n = \frac{2}{2}\int_{-1}^{1} f(t)\cos(n\omega t)\,dt = 2\int_0^1 f_{max}(1-t)\cos(n\cdot\frac{2\pi}{2}\cdot t)\,dt = \frac{2f_m}{(n\pi)^2}\cos(n\pi t)\Big|_1^0 \end{cases}$$

$$= \begin{cases} 0 & n = \text{even} \\ \frac{4f_{max}}{(n\pi)^2} & n = \text{odd} \end{cases}$$

☆ Plot is in Next page.

11.9. (a) For equal masses and identical springs, we have:

$$\begin{cases} m\ddot{x}_1 = -2Kx_1 + Kx_2 \\ m\ddot{x}_2 = Kx_1 - 2Kx_2 \end{cases}$$

Suppose $\begin{cases} \xi_1 = \frac{x_1+x_2}{2} \\ \xi_2 = \frac{x_1-x_2}{2} \end{cases}$

$$\Rightarrow \begin{cases} m\ddot{\xi}_1 = -K\xi_1 \\ m\ddot{\xi}_2 = -3K\xi_2 \end{cases}$$

$\Rightarrow$ ξ_1 and ξ_2 are uncoupled.

(b) $\xi_1 = A_1\cos(\omega_1 t - \delta_1)$ $\qquad \omega_1 = \sqrt{\frac{K}{m}}$

$\xi_2 = A_2\cos(\omega_2 t - \delta_2)$ $\qquad \omega_2 = \sqrt{\frac{3K}{m}}$

$$\Rightarrow \begin{cases} x_1 = \xi_1 + \xi_2 = A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2) \\ x_2 = \xi_1 - \xi_2 = A_1\cos(\omega_1 t - \delta_1) - A_2\cos(\omega_2 t - \delta_2) \end{cases}$$

11.12. (a) $\begin{cases} m\ddot{x}_1 = -Kx_1 - \beta m(\dot{x}_1 - \dot{x}_2) \\ m\ddot{x}_2 = Kx_2 - \beta m(\dot{x}_2 - \dot{x}_1) \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + \frac{K}{m}x_1 = 0 \\ \ddot{x}_2 + \beta(\dot{x}_2 - \dot{x}_1) + \frac{K}{m}x_2 = 0 \end{cases}$

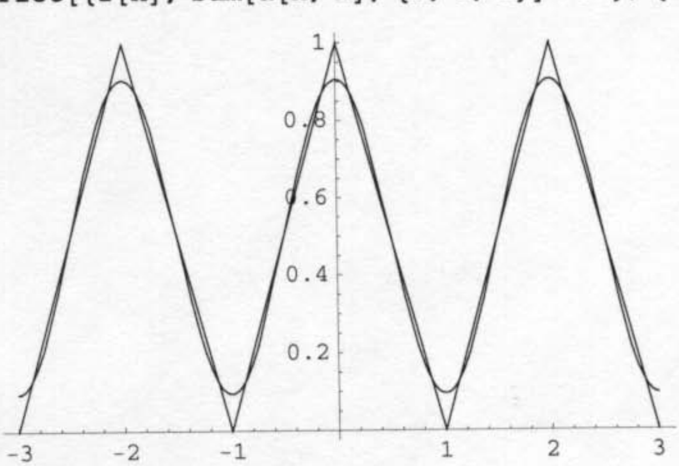
$\Rightarrow \ddot{\vec{x}} + \beta D\dot{\vec{x}} + \omega_0^2 x = 0$, $\quad D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\quad \omega_0^2 = \frac{K}{m}.$

```
In[4]:= (*This  is the plot of Homework #5 .49, although it is not required to work out
        the coefficients by computer, it is best to draw it by computer.*)
        f[x_] := (If[Mod[x, 2] == 0, Return[1]]; Abs[1 - Mod[Abs[x], 2]]) ;

In[5]:= a[x_, m_] := 4 / ((2 m + 1) * Pi) ^ 2 * Cos[(2 m + 1) * Pi * x];

In[6]:= a0 = 0.5;

In[7]:= Plot[{f[x], Sum[a[x, n], {n, 0, 0}] + a0}, {x, -3, 3}]
```

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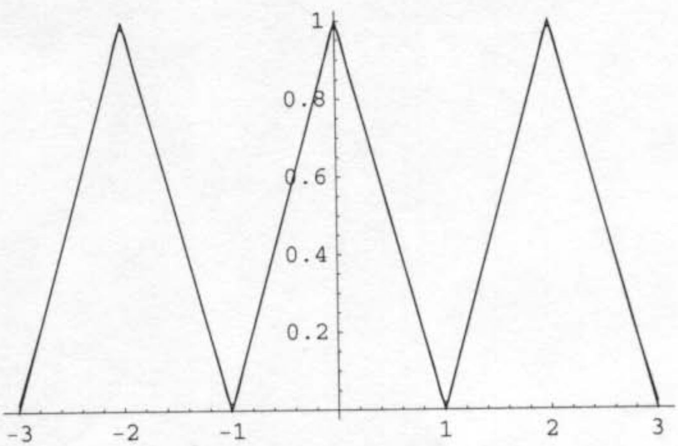
Out[7]= - Graphics -

In[8]:= Plot[{f[x], Sum[a[x, n], {n, 0, 5}] + a0}, {x, -3, 3}]
```

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Out[8]= - Graphics -
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11.12 (continue) (b) Suppose $\vec{x} = \mathrm{Re}\, z(t)$, $z(t) = a e^{rt}$

$$\Rightarrow (r^2 + \beta D r + \omega_0^2) a = 0$$

$$\Rightarrow \begin{bmatrix} \beta r + r^2 + \omega_0^2 & -\beta r \\ -\beta r & \beta r + r^2 + \omega_0^2 \end{bmatrix} a = 0$$

$$\Rightarrow \det \begin{bmatrix} \beta r + r^2 + \omega_0^2 & -\beta r \\ -\beta r & \beta r + r^2 + \omega_0^2 \end{bmatrix} = (r^2 + \omega_0^2)(r^2 + 2r\beta + \omega_0^2) = 0$$

$$\Rightarrow r_1 = i\omega_0, \quad r_2 = -\beta + i\sqrt{\omega_0^2 - \beta^2} \quad (\text{For } \beta < \omega_0)$$

(c) 1° $r = i\omega_0 \Rightarrow r^2 + \omega_0^2 + \beta r = \beta r$

$$\Rightarrow \begin{bmatrix} \beta r & -\beta r \\ -\beta r & \beta r \end{bmatrix} a = 0 \Rightarrow a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

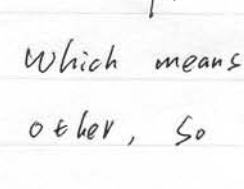
Which means that cart 1 & 2 are in phase, they are relatively stable to each other, so viscous drag is zero

2° $r = -\beta + i\sqrt{\omega_0^2 - \beta^2} \Rightarrow r^2 + \omega_0^2 + \beta r = -\beta r$

$$\Rightarrow \begin{bmatrix} -\beta r & -\beta r \\ -\beta r & -\beta r \end{bmatrix} a = 0 \Rightarrow a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Which means that carts' motion are out of phase, viscous drag supplies the damping force, so the motion is damping.

11.14



(a) For ϕ is very small, $x_i = L\phi_i$

$$\Rightarrow T = \frac{1}{2} m L^2 (\dot{\phi}_1^2 + \dot{\phi}_2^2)$$

$$U = \frac{1}{2} k L^2 (\phi_1 - \phi_2)^2 + mgL(1 - \cos\phi_1) + mgL(1 - \cos\phi_2)$$

$$\doteq \frac{1}{2} k L^2 (\phi_1 - \phi_2)^2 + \frac{1}{2} mgL(\phi_1^2 + \phi_2^2)$$

$$\Rightarrow \mathcal{L} = T - U$$

$$\Rightarrow \begin{cases} \dfrac{\partial \mathcal{L}}{\partial \phi_1} = -mgL\phi_1 + kL^2(\phi_2 - \phi_1) \\ \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = mL^2 \dot{\phi}_1 \end{cases} \quad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial \phi_2} = -mgL\phi_2 + kL^2(\phi_1 - \phi_2) \\ \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = mL^2 \dot{\phi}_2 \end{cases}$$

$$\Rightarrow \ddot{\phi}_1 = -\frac{g}{L}\phi_1 + \frac{k}{m}(\phi_2 - \phi_1); \quad \ddot{\phi}_2 = -\frac{g}{L}\phi_2 + \frac{k}{m}(\phi_1 - \phi_2)$$

(b) $\begin{cases} \ddot{\phi}_1 = -\frac{g}{L}\phi_1 + \frac{k}{m}(\phi_2 - \phi_1) \\ \ddot{\phi}_2 = -\frac{g}{L}\phi_2 + \frac{k}{m}(\phi_1 - \phi_2) \end{cases}$

$\Rightarrow M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $K = \begin{bmatrix} \omega_0^2 + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} \end{bmatrix}$, $\omega_0^2 = \frac{g}{L}$

$\Rightarrow [K - \omega^2 M] a = 0$

$\Rightarrow \det[K - \omega^2 M] = 0$

$\Rightarrow \omega_1 = \omega_0$, $\omega_2 = \sqrt{\omega_0^2 + 2k/m}$

For $\omega_1 = \omega_0$, we have

$$\begin{bmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} a = 0 \Rightarrow a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

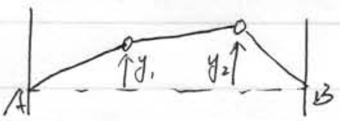
So the motion of balls are in phase, the spring is unstretched.

For $\omega_2 = \sqrt{\omega_0^2 + 2k/m}$, we have

$$\begin{bmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{bmatrix} a = 0 \Rightarrow a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So the motion of balls are out of phase, the spring is stretched.

11.24



For the displacement is very small, we know the total increase of the length of strings as:

$$\Delta d = \sqrt{L^2 + y_1^2} + \sqrt{L^2 + y_2^2} + \sqrt{L^2 + (y_1 - y_2)^2} - 3L$$

$$= \frac{1}{2L}[y_1^2 + y_2^2 + (y_1 - y_2)^2]$$

$$\Rightarrow U = T\Delta d = \frac{T}{2L}[y_1^2 + y_2^2 + (y_1 - y_2)^2]$$

$$T = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2)$$

$$\Rightarrow L = T - U = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - \frac{T}{2L}[y_1^2 + y_2^2 + (y_1 - y_2)^2]$$

$$\Rightarrow \begin{cases} m\ddot{y}_1 = -\frac{T}{2L}(4y_1 - 2y_2) = -\frac{T}{L}(2y_1 - y_2) \\ m\ddot{y}_2 = -\frac{T}{L}(2y_2 - y_1) \end{cases}$$

$$\Rightarrow M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} , \quad K = \begin{bmatrix} 2T/L & -T/L \\ -T/L & 2T/L \end{bmatrix}$$

$$\Rightarrow [K - w^2 M] a = 0$$

$$\Rightarrow \det [K - w^2 M] = 0$$

$$\Rightarrow (m w^2 - 2T/L)^2 - (T/L)^2 = 0$$

$$\Rightarrow w_1 = \frac{T}{mL}, \quad w_2 = \frac{3T}{8mL}$$

For $w_1 = \frac{T}{mL}$, we have

$$\begin{bmatrix} T/L & -T/L \\ -T/L & T/L \end{bmatrix} a = 0 \quad \Rightarrow a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which means the motions of the two masses are in phase

For $w_2 = \frac{3T}{mL}$, we have

$$\begin{bmatrix} -T/L & -T/L \\ -T/L & -T/L \end{bmatrix} a = 0 \quad \Rightarrow a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which means the motions of the two masses are out of phase.