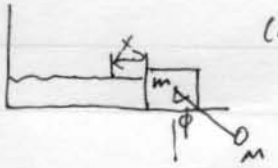
PHYS 410 Homework #9

7.31



(a) $X_M = x + L\sin\phi$, $\quad y_m = -L\cos\phi$ $\Rightarrow$ $T_m = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) = \frac{1}{2}m(\dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2)$, $\quad V_m = -mgL\cos\phi$

$$\Rightarrow L = T_M + T_m - (V_M + V_m)$$
$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}mL^2\dot{\phi}^2 + mL\dot{x}\dot{\phi}\cos\phi + mgL\cos\phi - \frac{1}{2}kx^2$$

$$\Rightarrow \begin{cases} \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + mL\dot{\phi}\cos\phi \\ \frac{\partial L}{\partial x} = -kx \end{cases} \Rightarrow (M+m)\ddot{x} + mL\ddot{\phi}\cos\phi - mL\dot{\phi}^2\sin\phi = -kx$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\phi}} = mL^2\dot{\phi} + mL\dot{x}\cos\phi \\ \frac{\partial L}{\partial \phi} = -mL\dot{x}\dot{\phi}\sin\phi - mgL\sin\phi \end{cases} \Rightarrow mL^2\ddot{\phi} + mL\ddot{x}\cos\phi - mL\dot{x}\dot{\phi}\sin\phi = -mL\dot{x}\dot{\phi}\sin\phi - mgL\sin\phi$$
$$\Rightarrow L\ddot{\phi} + \ddot{x}\cos\phi = -g\sin\phi$$

(b) If both x and ϕ are very small, we get:

$$\begin{cases} (M+m)\ddot{x} + mL\ddot{\phi} = -kx \\ L\ddot{\phi} + \ddot{x} = -g\phi \end{cases}$$

7.41 $L = \frac{1}{2}m[\rho^2\omega^2 + (2k\rho\dot{\rho})^2 + \dot{\rho}^2] - mgk\rho^2$

$$= \frac{1}{2}m[\omega^2\rho^2 + 4k^2\dot{\rho}^2\rho^2 + \dot{\rho}^2] - mgk\rho^2$$
$$\Rightarrow \frac{\partial L}{\partial \dot{\rho}} = 4mk^2\dot{\rho}\rho^2 + m\dot{\rho}$$
$$\frac{\partial L}{\partial \rho} = m\omega^2\rho + 4mk^2\dot{\rho}^2\rho - 2mgk\rho$$
$$\Rightarrow m\ddot{\rho} + 4k^2m\ddot{\rho}\rho^2 + 8mk^2\dot{\rho}^2\rho = m\omega^2\rho + 4mk^2\dot{\rho}^2\rho - 2mgk\rho$$
$$\Rightarrow (1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\dot{\rho}^2\rho = (\omega^2 - 2gk)\rho \quad - (2.1)$$

When the bead is in equilibrium, we get $\dot{\rho} = \ddot{\rho} = 0$, so:

$$(\omega^2 - 2gk)\rho = 0$$

1°. $\rho = 0$, from (2.1), we neglect high order terms and suppose the bead moves a small distance from $\rho = 0$, we could approximately get:

$$\ddot{\rho} = (\omega^2 - 2gk)\rho. \quad (\rho \ll 1)$$

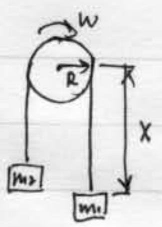
From the equation above, we could find that, when $\omega^2 > 2gk$, the equilibrium isn't stable; when $\omega^2 < 2gk$, the equilibrium is stable.

2° $\omega^2 = 2gk$. From (2.1), we get that

$$\ddot{\rho} = -\frac{4k^2\dot{\rho}^2}{1+4k^2\rho^2}\cdot\rho$$

So, the equilibrium is always stable, and the equilibrium point could be everywhere in the wire.

13.3



$$W = \frac{\dot{x}}{R} \Rightarrow T_M = \frac{1}{2} I_M \omega^2 = \frac{1}{2}\cdot\frac{1}{2}MR^2\cdot\left(\frac{\dot{x}}{R}\right)^2 = \frac{1}{4}M\dot{x}^2$$

$$T_m = \frac{1}{2}(m_1+m_2)\dot{x}^2$$

$$V = -m_1 g x - m_2 g(L-x)$$

$$\Rightarrow L = T - V = \frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{x}^2 + (m_1-m_2)gx + m_2 gL$$

$$\Rightarrow P = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2+\frac{1}{2}M)\dot{x}$$

$$\Rightarrow H = P\dot{x} - L = \frac{P^2}{2(m_1+m_2+\frac{1}{2}M)} - (m_1-m_2)gx - m_2 gL$$

$$\Rightarrow \begin{cases} \dot{x} = \frac{\partial H}{\partial P} = P/(m_1+m_2+\frac{1}{2}M) \\ \dot{P} = -\frac{\partial H}{\partial q} = (m_1-m_2)g \end{cases}$$

13.5

$$\dot{z} = c\dot{\phi}, \quad \dot{\rho} = 0$$

$$\Rightarrow T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) = \frac{1}{2}m(R^2+c^2)\dot{\phi}^2$$

$$U = mgz = mgc\phi$$

$$\Rightarrow L = T - U = \frac{1}{2}m(R^2+c^2)\dot{\phi}^2 - mgc\phi$$

$$\Rightarrow P_\phi = m(R^2+c^2)\dot{\phi}$$

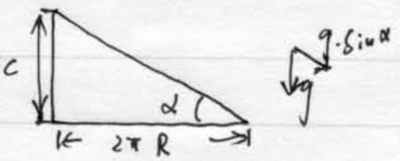
$$\Rightarrow H = T + U = \frac{P_\phi^2}{2m(R^2+c^2)} + mgc\phi$$

$$\Rightarrow \begin{cases} \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m(R^2+c^2)} \\ \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = -mgc \end{cases}$$

$$\Rightarrow \ddot{\phi} = -\frac{gc}{R^2+c^2}$$

$$\Rightarrow \ddot{z} = c\ddot{\phi} = -\frac{gc^2}{R^2+c^2} \quad -(4.1)$$

Let's unwrap the helix, and show it in 2D space;



$\Rightarrow$ A slop with $tg\alpha = \frac{C}{R}$

$(4.1) \Rightarrow \quad \ddot{z} = -g \cdot \sin^2\alpha$

If we analyze the forces felt by the bead, we find that it is accelerated by the gravity along the helix line, which is obvious in the 2D picture. The acceleration along the slope is $-g \cdot \sin\alpha$; while its projection in z axis, which is the accelration in z direction is $-g \cdot \sin\alpha \cdot \sin\alpha = -g \cdot \sin^2\alpha$.

If $R=0$, it turns to a free body falling problem, and (4.1) changes to:

$$\ddot{z} = -g.$$

13.7 (a) $L = T - V = \frac{1}{2}m(\dot{x}^2 + h'^2_{(x)} \cdot \dot{x}^2) - mgh(x)$

$$\Rightarrow P_x = \frac{\partial H}{\partial \dot{x}} = m(1+h'^2_{(x)})\dot{x}$$

$$\Rightarrow H = P_x \cdot \dot{x} + mgh(x) - \frac{1}{2}m(\dot{x}^2 + h'^2_{(x)}\dot{x}^2)$$

$$= \frac{P_x^2}{2(1+h'^2_{(x)})m} + mgh(x)$$

(b)

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{\cancel{2}(1+h'^2_{(x)})m} \\ \dot{P}_x = -\frac{\partial H}{\partial x} = \frac{P_x^2 h'(x) h''(x)}{(1+h'^2_{(x)})^2 m} - mgh' = mh'h''\dot{x}^2 - mgh' \end{cases}$$

$$\Rightarrow \ddot{x} = \frac{\dot{P}_x}{m(1+h'^2(x))} - \frac{2P_x \cdot h'h''\dot{x}}{(1+h'^2_{(x)})^2 m}$$

$$= \frac{h'h''\dot{x}^2 - gh'}{1+h'^2_{(x)}} - \frac{2h'h''\dot{x}^2}{1+h'^2_{(x)}}$$

$$= -\frac{gh' + h'h''\dot{x}^2}{1+h'^2(x)} \quad — (5.1)$$

In Netownian Mechanics, we know:

$$\ddot{S} = \frac{1}{m} \cdot \left(-\frac{\partial V}{\partial S}\right) = \frac{1}{m} \cdot \left(-\frac{mgh'(x)dx}{\sqrt{1+h'^2_{(x)}}\,dx}\right) = -\frac{gh'(x)}{\sqrt{1+h'^2_{(x)}}}$$

$$\dot{S} = \sqrt{1+h'^2_{(x)}}\,\dot{x}$$

$$\Rightarrow \ddot{S} = \sqrt{1+h'^2_{(x)}}\,\ddot{x} + \frac{\dot{x}^2 h'(x) h''(x)}{\sqrt{1+h'^2_{(x)}}}$$

$$\Rightarrow \sqrt{1+h'^2(x)}\ \ddot{x} = -\frac{g\,h'(x) + \dot{x}^2 h'(x) h''(x)}{\sqrt{1+h'^2(x)}}$$

$$\Rightarrow \ddot{x} = -\frac{g h' + \dot{x}^2 h' h''}{1 + h'^2(x)} \quad - (5.2)$$

we find (5.1) coincides with (5.2)

13.10 $\quad F = -kx\hat{x} + K\hat{y}$

$$\Rightarrow U = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r} = \tfrac{1}{2}kx^2 - Ky$$

$$\Rightarrow L = T - U = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}kx^2 + Ky$$

$$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\Rightarrow H = p_x\dot{x} + p_y\dot{y} - L = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}kx^2 - Ky = \frac{p_x^2+p_y^2}{2m} + \tfrac{1}{2}kx^2 - Ky$$

$$\Rightarrow \begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = K \end{aligned}$$

$$\Rightarrow \begin{cases} \ddot{y} = \frac{K}{m} \\ y = \frac{1}{2}\cdot\frac{K}{m}t^2 + v_{y_0}t + y_0 \end{cases}$$

13.12. $\quad L = T - U, \quad T = \tfrac{1}{2}m[\dot{x}^2 + (\omega x)^2]$

$$\Rightarrow L = \tfrac{1}{2}m[\dot{x}^2 + (\omega x)^2]$$

$$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\Rightarrow H = p_x\dot{x} - L = \tfrac{1}{2}m[\dot{x}^2 - (\omega x)^2] \neq T + U$$

13.14. From the textbook, we know:

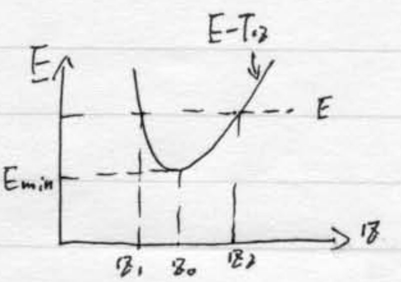
$$T = \tfrac{1}{2}m[(c^2+1)\dot{z}^2 + (cz\dot{\phi})^2]$$

$$\Rightarrow \begin{cases} p_z = \frac{\partial T}{\partial \dot{z}} = m(c^2+1)\dot{z}, \quad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mc^2z^2\dot{\phi}, \quad \dot{p}_\phi = 0 \\ H = \frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2z^2}\right] + mgz \end{cases}$$

$$\Rightarrow \begin{cases} \text{when } \dot{z} = 0, \quad p_z = 0 \\ \text{when } p_z = 0, \quad H = \frac{p_\phi^2}{2mc^2z^2} + mgz = E \end{cases}$$

$$\Rightarrow H \text{ has a minimum when } \left.\frac{\partial H}{\partial z}\right|_{z_0} = 0 \Rightarrow z_0 = \left(\frac{p_\phi^2}{m^2c^2g}\right)^{1/3}$$

$$\Rightarrow E_{min} = \frac{mg}{2}\cdot\left(\frac{p_\phi^2}{mc^2g}\right)^{1/3}$$



1° $E < E_{min}$, this system has no physical meaning

2° $E = E_{min}$, there is only one equilibrium point in z axis, which corresponds to a motion in a circle.

3° $E > E_{min}$, there is a range of equilibrium points, with two boundary points at z_1, z_2

While p_ϕ is const, $\dot{\phi}$ doesn't change its direction, that means the motion in polar coordinate is always in the same direction.

13.18. (a) $L = \frac{1}{2} m \dot{\vec{r}}^2 - qV + q\vec{A}\cdot\dot{\vec{r}}$

$$\Rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

$$\Rightarrow H = \vec{p}\cdot\dot{\vec{r}} - L = \frac{(\vec{p} - q\vec{A})^2}{2m} + qV$$

(b) $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x - qA_x}{m}$, $\dot{p}_x = -\frac{\partial H}{\partial x} = q\left(\sum_{i=1}^{3} \dot{r}_i \frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right)$

$$\Rightarrow m\ddot{x} = \dot{p}_x - q\frac{dA_x}{dt} = q\left(\sum_i \dot{r}_i \frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right) - q\left(\sum_i \frac{\partial A_x}{\partial r_i}\dot{r}_i + \frac{\partial A_x}{\partial t}\right)$$

$$= q\left[-\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$= q E_x + q(\vec{v}\times\nabla\times\vec{A})_x$$

$$= q E_x + q(\vec{v}\times\vec{B})_x$$

Similarly, we get $m\ddot{y}$ and $m\ddot{z}$ which are the corresponding component of $q\vec{E}$ and $q\vec{v}\times\vec{B}$, put them together, we have:

$$m\ddot{\vec{r}} = q[\vec{E} + (\vec{v}\times\vec{B})]$$

13.20 (a) $U(r) = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r} = -\vec{F}\cdot\vec{r}$

$$L = \frac{1}{2} m\dot{\vec{r}}^2 + \vec{F}\cdot\vec{r}$$

$$\Rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

$$\Rightarrow H = \vec{p}\cdot\dot{\vec{r}} - L = \frac{\vec{p}^2}{2m} - \vec{F}\cdot\vec{r}$$

(b) If $\vec{F} = F\hat{x}$, we get:

$$H = \frac{p_x^2 + p_y^2}{2m} - Fx$$

$$\Rightarrow \dot{p}_y = -\frac{\partial H}{\partial y} = 0$$

$\Rightarrow$ y is ignorable.

(c) $\vec{F} = F_x\hat{x} + F_y\hat{y}$

$$\Rightarrow H = \frac{p_x^2 + p_y^2}{2m} - F_x\cdot x - F_y\cdot y$$

$$\Rightarrow \begin{cases} \dot{p}_x = -\dfrac{\partial H}{\partial x} = F_x \neq 0 \\ \dot{p}_y = -\dfrac{\partial H}{\partial y} = F_y \neq 0 \end{cases}$$

$\Rightarrow$ $x \cdot y$ are not ignorable.