PHYS 410 Homework # 8

6.1. On a spherical surface, the longitude and latitude lines are orthogonal.

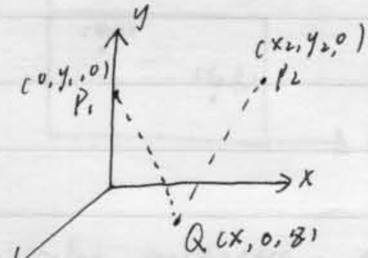
For a small change in position, we have the change in longitude and latitude:

$$\Delta d_{lo} = R\cdot d\theta \quad , \quad \Delta d_{la} = R\cdot \sin\theta \cdot d\phi$$

$$\Rightarrow ds = \sqrt{\Delta d_{lo}^2 + \Delta d_{la}^2} = R\, d\theta \sqrt{1+\sin^2\theta \frac{d\phi}{d\theta}}$$

$$\Rightarrow L = \int_{P_1}^{P_2} ds = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta \frac{d\phi}{d\theta}} \cdot d\theta$$

6.3



$$L(PQP_2) = \sqrt{x^2+y^2+z^2} + \sqrt{(x_2-x)^2+y_2^2+z^2}$$

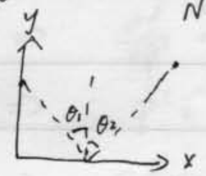
$$\Rightarrow T = L/c$$

According to Fermat's principle, we get:

$$\frac{\partial T}{\partial x} = 0 \quad , \quad \frac{\partial T}{\partial z} = 0 .$$

$$\frac{\partial T}{\partial z} = 0 \Rightarrow z\cdot\left(\frac{1}{\sqrt{x^2+y_1^2+z^2}} + \frac{1}{\sqrt{(x_2-x)^2+y_2^2+z^2}}\right) = 0 \Rightarrow z = 0 \qquad -(1)$$

$$\frac{\partial T}{\partial x} = 0 \Rightarrow \frac{x}{\sqrt{x^2+y_1^2+z^2}} + \frac{x-x_2}{\sqrt{(x_2-x)^2+y_2^2+z^2}} = 0 \qquad -(2)$$

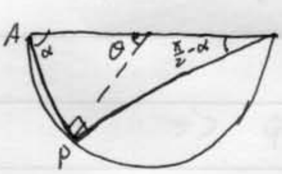


According to (1), when $z=0$, we get from (2):

$$\sin\theta_1 = \sin\theta_2$$

$$\Rightarrow \theta_1 = \theta_2$$

6.5



According to General Geometric relation, we know $\angle APB = \frac{\pi}{2}$.

And $\alpha = \frac{\pi-\theta}{2}$

$$\Rightarrow AP = AB\cdot\sin\left(\tfrac{\pi}{2}-\alpha\right) = AB\cos\tfrac{\alpha}{2} = AB\sin\tfrac{\theta}{2} = 2R\sin\tfrac{\theta}{2}$$

$$BP = AB\cdot\sin\alpha = 2R\cdot\cos\tfrac{\theta}{2}$$

$$\Rightarrow APB = 2R\cdot\left(\sin\tfrac{\theta}{2}+\cos\tfrac{\theta}{2}\right) = 2\sqrt{2}\,\sin\left(\tfrac{\theta}{2}+\tfrac{\pi}{4}\right)R$$

$\Rightarrow (APB)_{max} = 2\sqrt{2}R$ when $\theta = \frac{\pi}{2}$, which is the real path.

6.7

In cylinder coordinates, when R is const, the infinitesimal change in position corresponds to $ds = \sqrt{R^2 d\phi^2 + (dz)^2} = dz\sqrt{1+R^2\phi'^2_{(z)}} = f(\phi', z)\, dz$

$$\Rightarrow L = \int_{P_1}^{P_2} ds = \int_{z_1}^{z_2} f(\phi', z)\, dz$$

According to Euler - Lagrange Eqns:

$$\frac{\partial f}{\partial \phi} = \frac{d}{dz}\frac{\partial f}{\partial \phi'}$$

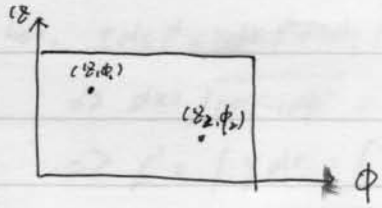
$$\Rightarrow \frac{\partial f}{\partial \phi'} = \frac{R^2\phi'}{\sqrt{R^2\phi'^2+1}} = const$$

$$\Rightarrow \phi'_{(z)} = const$$

$$\Rightarrow \phi_{(z)} = az+b \quad , \quad a, b \text{ are consts, and determined by end points}$$

$$\Rightarrow \phi_{(z)} = \frac{\phi_1-\phi_2}{z_1-z_2}\cdot z + \frac{\phi_2 z_1 - \phi_1 z_2}{z_1-z_2} \quad , \text{ which is unique.}$$

If we cut the cylinder into a plane, which is without wrap, we get:



So the shortest path is, of course, the line connecting the end points.

(Note: we suppose ϕ is unique. If $\phi \neq \phi+2n\pi$, you will get different answer)
in a point — and — corresponds to same point

6.18. In polar coordinates, the infintesimal disturbance is:

$$\Delta d = \sqrt{(dr)^2+r^2(d\phi)^2} = \sqrt{1+r^2\phi'^2}\,dr = f\,dr$$

$$\Rightarrow \frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) = \frac{\partial f}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial f}{\partial \phi'} = \frac{r^2\phi'}{\sqrt{1+r^2\phi'^2}} = const$$

$$\Rightarrow \phi' = \frac{c}{r\cdot\sqrt{r^2-c^2}} \quad , \quad c = const$$

$$\Rightarrow \text{Suppose } c/r = \cos u$$

$$\int d\phi = \int \frac{\cos u}{r\cdot \sin u}\,dr$$

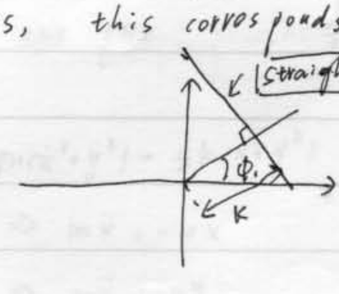
$$= \int \frac{\cos^2 u}{\sin u}\cdot\frac{\sin u}{\cos^2 u}\,du$$

$$\Rightarrow \phi - \phi_0 = \arccos(k/r) - \arccos(k/r_0)$$

$$\Rightarrow \arccos k/r = \phi - \phi_1 \quad , \quad \phi_1 = const, \text{ determined by end points.}$$

$$\Rightarrow k = r\cos(\phi-\phi_1)$$

In polar coordinates, this corresponds to a straight line, which is perpendicular to $\phi=\phi_1$;



6.22 First, Let's find a const when $f(y', y, x)$ doesn't depend on x:

$$\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''$$

According to Euler - Lagrange Eq:

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y'}$$

$$\Rightarrow \frac{df}{dx} = \frac{d}{dx} (y' \frac{\partial f}{\partial y'})$$

$$\Rightarrow f - y' \frac{\partial f}{\partial y'} = \text{const} \quad - (3)$$

Then, $(ds)^2 = (dx)^2 + (dy)^2$

$$\Rightarrow dx = \sqrt{(ds)^2 - (dy)^2} = \sqrt{1 - (\frac{dy}{ds})^2}\, ds$$

$$\Rightarrow S = \int y\, dx = \int_0^l y \sqrt{1 - (\frac{dy}{ds})^2}\, ds$$

$$\Rightarrow f = y\sqrt{1 - y'^2} \quad - (4)$$

$$(3),(4) \Rightarrow y\sqrt{1-y'^2} + y' \cdot \frac{y y'}{\sqrt{1-y'^2}} = R \quad , \qquad R \text{ is a const}$$

$$\Rightarrow y' = \sqrt{1 - (\frac{y}{R})^2}$$

$$\Rightarrow ds = \frac{dy}{\sqrt{1-(\frac{y}{R})^2}}$$

$$\Rightarrow y = R\sin(S/R)$$

$$y = 0,\ S = l \Rightarrow l/R = n\pi \qquad (n = 1, 2 \cdots)$$

$$dx = \sqrt{1-y'^2}\, ds \Rightarrow \underset{(l)}{x} = \int_0^l \sqrt{1-y'^2}\, ds = 2n\,[R - R\cos(l/R)]$$

So There're n similary parts in xy plane, Let's just discuss 1st part

$$x(s) = \int_0^s \sqrt{1-y'^2}\, ds = R - R\cos(S/R)$$

$$\Rightarrow (x-R)^2 + y^2 = R^2 \quad \Rightarrow S_1 = \pi R^2$$

$$\Rightarrow S = n\pi R^2 = n\pi \cdot \frac{l^2}{n^2\pi^2} = \frac{l^2}{n\pi}$$

$$\Rightarrow S_{max} = \frac{l^2}{\pi}, \quad n = 1$$

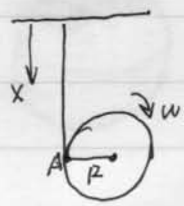
$\Rightarrow$ Only one part : $(x - \frac{l}{\pi})^2 + y^2 = (\frac{l}{\pi})^2$, which is a semicircle.

7.3. $L = T - U = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$

$$\begin{cases} \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -kx \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = -ky \end{cases}$$

The equations of motion tell us the oscillation frequency in x and y axis are same. The physical picture of this motion is a 2D linear oscillator, whose motion in x and y axes satisfy the equations above. The exact solution depends on initial conditions.

7.14



Suppose Yoyo roll about its origin a cycle, then the contacting point of the Yoyo with the vertide line is still the point A on Yoyo, so we got

$$\frac{2\pi R}{\dot{x}} = \frac{2\pi}{w}$$

$$\Rightarrow \dot{x} = Rw$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Iw^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\cdot\frac{1}{2}mR^2\cdot\frac{\dot{x}^2}{R^2} = \frac{3}{4}m\dot{x}^2$$

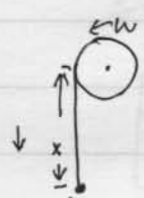
$$L = T - V = \frac{3}{4}m\dot{x}^2 + mgx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\Rightarrow \frac{3}{2}m\ddot{x} = mg$$

$$\Rightarrow \ddot{x} = \frac{2}{3}g$$

7.18.



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Iw^2$$

$$w = \frac{\dot{x}}{R}$$

$$\Rightarrow L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\cdot\frac{I}{R^2}\dot{x}^2 + mgx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = mg$$

$$\Rightarrow \left(m + \frac{I}{R^2}\right)\ddot{x} = mg$$

$$\Rightarrow \ddot{x} = \frac{mg}{m + I/R^2}$$

7.23. For the system of Large and small carts

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{x})^2 - \frac{1}{2}kx^2 - U(X)$$

where $U(X)$ is the external potential forced on the large cart to make its oscillation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\Rightarrow m\ddot{x} + m\ddot{X} = -kx$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = A w^2 \cos wt$$