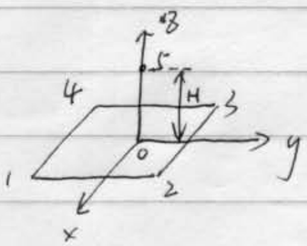
Solutions of PHYS 410 Homework #7

10.3



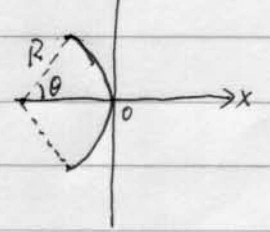
First just take particles 1.2.3.4 in xy plane, and from the symmetry, we could take it as a particle in origin O, with four times mass of individual particle.

Then, the CM of 5 and this effective particle is:

$$z_{cm} = \frac{mH}{5m} = \frac{H}{5}.$$

So $CM = (0, 0, \frac{H}{5})$

10.8



From symmetry, we could find CM only has x component.

$$X_{cm} = -\frac{\int_{-\theta}^{\theta}(R - R\cos\theta')R\,d\theta'}{\int_{-\theta}^{\theta}R\,d\theta'} = R\left[\frac{\sin\theta}{\theta} - 1\right] \quad - ①$$

$$2R\cdot\theta = L \quad - ②$$

$$①.② \Rightarrow X_{cm} = R\left[\frac{\sin(L/2R)}{L/2R} - 1\right] \quad - ③$$

$$CM = \left(R\left[\frac{\sin(L/2R)}{L/2R} - 1\right], 0\right)$$

when $R \to \infty$, $\sin(L/2R)/(L/2R) \to 1 \Rightarrow X_{cm} \Rightarrow 0$, which is the case that the wire is not bent.

when $L = 2\pi R \Rightarrow \theta = \pi \Rightarrow X_{cm} = -R$, which is obvious from symmetry view.

10.14. (Note: flywheel is taken as a thin disk)

$$I_W = M_W \frac{\rho\int_0^{2\pi}d\theta\int_0^{R_W} r^2\cdot r\,dr}{\rho\int_0^{2\pi}d\theta\int_0^{R} r\,dr} = \frac{1}{2}M_W R_W^2 = \frac{1}{2}\times(0.1)^2\times 10 = 5\times10^{-2}\ kg\cdot m^2$$

$$I_S = M_S \frac{\rho\cdot\int_0^{\pi}\sin^2\theta\sin\theta\,d\theta\cdot\int_{R_{S2}}^{R_{S1}} r^2\cdot r^2\,dr}{\rho\int_0^{\pi}\sin\theta\,d\theta\left(\int_0^{R_{S1}}r^2\,dr - \int_0^{R_{S2}}r^2\,dr\right)} = \frac{2}{5}M_S \frac{R_{S1}^5 - R_{S2}^5}{R_{S1}^3 - R_{S2}^3} = 1.23\times10^5\ kg\cdot m^2$$

(1) From the conversation of angular momentum, we know:

$$I_W\cdot\Delta W_W = I_S\cdot\Delta W_S$$

$$\Rightarrow \Delta W_S = \frac{I_W}{I_S}\cdot\Delta W_W \quad -(14.1)$$

$$t = \frac{\Delta\theta}{\Delta W_S} = \frac{\Delta\theta}{\Delta W_W}\cdot\frac{I_S}{I_W} = 68.3\ min \qquad (\text{Note: } \Delta\theta = 10^\circ = \tfrac{1}{36}\ \text{round})$$

(2) $T = \frac{1}{2} I_s w_s^2 + \frac{1}{2} I_w W_w^2$

From (14.1), we know that

$T = \frac{1}{2} I_w \cdot \frac{I_w}{I_s} \cdot W_w^2 + \frac{1}{2} I_w W_w^2$

Because $\frac{I_w}{I_s} \sim 10^{-7}$, we get:

$T \doteq \frac{1}{2} I_w W_w^2 = \frac{1}{2} \times 5\times10^{-2} \times (\frac{2\pi\times10^3}{60})^2 \doteq 274 J$

10.19 ① $\vec{w} = (w_x, w_y, w_z)$, $\vec{r} = (x, y, z)$

$\vec{w}\times\vec{r} = (w_y z - w_z y, w_z x - w_x z, w_x y - w_y x)$

$[\vec{r}\times(\vec{w}\times\vec{r})]_x = y(w_x y - w_y x) - z(w_z x - w_x z) = (y^2+z^2) w_x - xy w_y - xz w_z$

Similarly, we could find y and z components of $\vec{r}\times(\vec{w}\times\vec{r})$, which are the same with (10.35)

② $\vec{r}\times(\vec{w}\times\vec{r}) = (\vec{r}\cdot\vec{r})\vec{w} - (\vec{r}\cdot\vec{w})\vec{r} = r^2\vec{w} - (\sum_i r_i w_i)\vec{r}$

$[\vec{r}\times(\vec{w}\times\vec{r})]_j = r^2 w_j - \sum_i (r_i w_i) r_j = (r^2 - r_j^2) w_j - \sum_{j\neq i} r_j r_i w_i$

which are the same form with (10.35)

10.21 There are several ways to prove this, I choose one using matrix method:

$\vec{L} = m[\vec{r}\times(\vec{w}\times\vec{r})]$, $L_i = m[(r^2 - r_i^2)w_i - \sum_{j\neq i} r_i r_j w_j] = m\sum_j (r^2\delta_{ij} - r_i r_j) w_j$ — (21.1)

In another way, $\vec{L} = \underline{\underline{I}}\cdot\vec{w}$ ($\underline{\underline{I}}$ is the tensor)

$L_i = \sum_j I_{ij} w_j$ (21.2)

Compare (21.1) and (21.2), we get:

$I_{ij} = m(r^2\delta_{ij} - r_i r_j)$

If the mass is distributed continuously, we get:

$I_{ij} = \int \rho (r^2\delta_{ij} - r_i r_j) dV$

10.33 (a) $T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$, $T = \sum_\alpha T_\alpha = \frac{1}{2}\sum_\alpha m_\alpha (\vec{w}\times\vec{r_\alpha})^2$

$(\vec{w}\times\vec{r_\alpha})^2 = (\vec{w}\times\vec{r_\alpha})\cdot(\vec{w}\times\vec{r_\alpha}) = \vec{w}\cdot[\vec{r_\alpha}\times(\vec{w}\times\vec{r_\alpha})] = \vec{w}\cdot[r_\alpha^2\vec{w} - (\vec{r_\alpha}\cdot\vec{w})\vec{r_\alpha}] = r_\alpha^2 w^2 - (\vec{r_\alpha}\cdot\vec{w})^2$

$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha [r_\alpha^2 w^2 - (\vec{r_\alpha}\cdot\vec{w})^2]$

(b) $\vec{L} = \sum_\alpha \vec{r}_\alpha \times (m_\alpha \vec{v}_\alpha) = \sum_\alpha \vec{r}_\alpha \times [m_\alpha (\vec{w} \times \vec{r}_\alpha)]$

$= \sum_\alpha m_\alpha [(\vec{r}_\alpha \cdot \vec{r}_\alpha)\vec{w} - (\vec{r}_\alpha \cdot \vec{w})\vec{r}_\alpha]$

$= \sum_\alpha m_\alpha [r_\alpha^2 \vec{w} - (\vec{r}_\alpha \cdot \vec{w})\vec{r}_\alpha]$

(c) $T = \sum_\alpha \frac{1}{2} m_\alpha (\vec{w} \times \vec{r}_\alpha)^2 = \frac{1}{2} \sum_\alpha m_\alpha (\vec{w} \times \vec{r}_\alpha) \cdot (\vec{w} \times \vec{r}_\alpha) = \frac{1}{2} \sum_\alpha \vec{w} \cdot m_\alpha \vec{r}_\alpha \times (\vec{w} \times \vec{r}_\alpha)$

$= \frac{1}{2} \vec{w} \cdot \vec{L} = \frac{1}{2} \vec{w} \cdot (\underline{\underline{I}} \cdot \vec{w})$

if write in matrix form, we get:

$T = \frac{1}{2} \tilde{w} \cdot \underline{\underline{I}} \cdot w$, where w is column matrix, $\tilde{w}$ is the transpose matrix of w.

(BTW: In fact, in Goldstein's Book, we just denote $T = \frac{1}{2} \vec{w} \cdot \underline{\underline{I}} \cdot \vec{w}$)

(d) if $\underline{\underline{I}}$ is about principal axes, we could write $\underline{\underline{I}}$ as:

$$\underline{\underline{I}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$T = \frac{1}{2} (w_1 \ w_2 \ w_3) \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{2} \sum_i \lambda_i w_i^2$$

10.35 (a)

$$\underline{\underline{I}} = \sum_\alpha m_\alpha \begin{bmatrix} r_\alpha^2 - x_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & r_\alpha^2 - y_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & r_\alpha^2 - z_\alpha^2 \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

(b) $|\underline{\underline{I}} - \lambda \underline{\underline{1}}| = 0 \Rightarrow \lambda_1 = 10ma^2, \lambda_2 = 7ma^2, \lambda_3 = 5ma^2$

Principal axes corresponds to eigenvectors of $\underline{\underline{I}}$

Corresponding $\lambda_1, \lambda_2, \lambda_3$, the principal axes are:

$\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = \frac{1}{\sqrt{2}}(0, 1, 1)$, $\vec{e}_3 = \frac{1}{\sqrt{2}}(0, 1, -1)$

10.40. In nonrelativistic mechanics, the change of frame doesn't affect the magnitude of a vector. So, if the magnitude of a vector is constant in one frame, it's also the case in all others. clearly, if there is no torque forced on the body, $\vec{L}$ is obviously constant in space frame, which means its magnitude is constant in all frames, so is T. To prove this again using Euler equations is just to impress you the effect of Euler equations.

(a)
$$\begin{cases} \lambda_1 \dot{w}_1 = (\lambda_2 - \lambda_3) w_2 w_3 \\ \lambda_2 \dot{w}_2 = (\lambda_3 - \lambda_1) w_3 w_1 \\ \lambda_3 \dot{w}_3 = (\lambda_1 - \lambda_2) w_1 w_2 \end{cases} \quad -(40.1)$$

Multiply $\lambda_i w_i$ on the ith equation of (40.1), and add all the equations of (40.1) together, we get:

$$\frac{1}{2}\frac{d}{dt}\left(\sum_i (\lambda_i w_i)^2\right) = w_1 w_2 w_3 \left[\lambda_1(\lambda_2-\lambda_3) + \lambda_2(\lambda_3-\lambda_1) + \lambda_3(\lambda_1-\lambda_2)\right] = 0$$

$$\Rightarrow \frac{d}{dt}|L|^2 = 0$$

(b) Multiply w_i on the ith equation of (40.1), and add all equations of (40.1), we get:

$$\frac{1}{2}\frac{d}{dt}\left(\sum_i \lambda_i w_i^2\right) = w_1 w_2 w_3 \left[\lambda_2 - \lambda_3 + \lambda_3 - \lambda_1 + \lambda_1 - \lambda_2\right] = 0$$

$$\Rightarrow \frac{d}{dt} T = 0$$

10.50 From (10.99), we get:

$$\vec{w} = (-\dot\phi \sin\theta)\vec{e}_1{}' + \dot\theta \vec{e}_2{}' + (\dot\psi + \dot\phi\cos\theta)\vec{e}_3$$

Because $\vec{e}_1{}' = \vec{e}_1 \cdot \cos\psi - \vec{e}_2 \cdot \sin\psi$, $\vec{e}_2{}' = \vec{e}_1 \cdot \sin\psi + \vec{e}_2 \cdot \cos\psi$

we could rewrite $\vec{w}$ as:

$$\vec{w} = -(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\vec{e}_1 + (\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\vec{e}_2 + (\dot\psi + \dot\phi\cos\theta)\vec{e}_3$$

$$T = \frac{1}{2}\lambda_1 w_1^2 + \frac{1}{2}\lambda_2 w_2^2 + \frac{1}{2}\lambda_3 w_3^2$$

$$= \frac{1}{2}\lambda_1(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)^2 + \frac{1}{2}\lambda_2(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)^2 + \frac{1}{2}\lambda_3(\dot\psi + \dot\phi\cos\theta)^2$$

10.51
$$E = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\psi + \dot\phi\cos\theta)^2 + MgR\cos\theta \quad -(51.1)$$

$$\lambda_1(\dot\phi\sin^2\theta) + \lambda_3(\dot\psi + \dot\phi\cos\theta)\cos\theta = L_z \quad -(51.2)$$

$$\lambda_3(\dot{\psi}+\dot{\phi}\cos\theta)=L_3 \qquad -(51.3)$$

Combine (51.1). (51.2) . (51.3), we get:

$$E=\frac{1}{2}\lambda_1\dot{\theta}^2+\frac{1}{2}\lambda_1\cdot\frac{(L_3-L_3\cos\theta)^2}{\lambda_1^2\sin^2\theta}+\frac{1}{2}\lambda_3\cdot\left(\frac{L_3}{\lambda_3}\right)^2+mgR\cos\theta$$

$$=\frac{1}{2}\lambda_1\dot{\theta}^2+U_{eff}(\theta)$$

$$\Rightarrow U_{eff}(\theta)=\frac{1}{2}\cdot\frac{(L_3-L_3\cos\theta)^2}{\lambda_1\sin^2\theta}+\frac{1}{2}\cdot\frac{L_3^2}{\lambda_3^2}+mgR\cos\theta$$