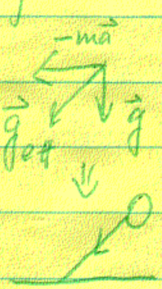
PHYS 410 Homework #6

9.1. The direction of string in the frame of the moving car should be in the direction of effective gravity. And effective gravity is the combination of true gravity and inertial force $-m\vec{a}$.



From the picture, we could clearly find why the ballon is in the direction forward.

Note: In this noninertial frame, buoyant force also directs in the direction as $\vec{g}_{eff}$.

9.3 (a) $d = d_0 - R_e = d_0(1 - R_e/d_0)$

$$\Rightarrow \vec{F} = -GM_m m\left(\frac{1}{d^2} - \frac{1}{d_0^2}\right)\hat{d} = -GM_m m\left(\frac{1}{d_0^2(1-R_e/d_0)^2} - \frac{1}{d_0^2}\right)\hat{d}$$

$$= -GM_m m \cdot \frac{\hat{d}}{d_0^2}\cdot\left(1 + \frac{2R_e}{d_0} - 1\right)$$

$$= -GM_m m \cdot \frac{2R_e}{d_0^3}\hat{d}$$

Clearly the direction of $\vec{F}$ is opposite to $\hat{d}$.

$$mg = \frac{GM_e m}{R_e^2}$$

$$\Rightarrow \frac{|\vec{F}|}{mg} = 2\cdot\left(\frac{R_e}{d_0}\right)^3 \cdot \frac{M_m}{M_e} = 2\times\left(\frac{6.37\times10^6}{3.84\times10^8}\right)^3\times\frac{7.35\times10^{22}}{5.98\times10^{24}} \doteq 1.12\times10^{-7}$$

(b) $d' = d_0 + R_e = d_0(1 + R_e/d_0)$

$$\Rightarrow \vec{F}' = -GM_m m\left(\frac{1}{d'^2} - \frac{1}{d_0^2}\right)\hat{d} = GM_m m\frac{\hat{d}}{d_0^2}\left(1 - \frac{1}{(1+R_e/d_0)^2}\right)$$

$$= GM_m m\frac{\hat{d}}{d_0^2}\cdot\frac{2R_e}{d_0} = GM_m m\frac{2R_e}{d_0^3}\hat{d}$$

$|\vec{F}'| = |\vec{F}|$, but $\vec{F}' = -\vec{F}$ by opposite direction.

9.8 (a) South near North pole: F_{cf}: south and up, but $(F_{cf})_s \gg (F_{cf})_u$

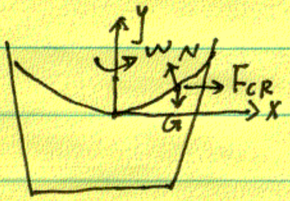
(b) east on the equator: F_{cf} : vertically up
F_{cor} : vertically up

(c) south across the equator: F_{cf} : vertically up
F_{cor} : zero

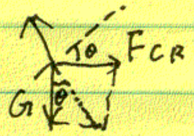
9.10 $(\frac{d\vec r}{dt})_{S_0} = (\frac{d\vec r}{dt})_S + \vec\Omega\times\vec r$

$$\Rightarrow (\frac{d^2 r}{dt})_{S_0} = (\frac{d\vec r}{dt})_{S_0}\left((\frac{d\vec r}{dt})_S + \vec\Omega\times\vec r\right)$$
$$= (\frac{d}{dt})_S\left((\frac{d\vec r}{dt})_S + \vec\Omega\times\vec r\right) + \vec\Omega\times\left((\frac{d\vec r}{dt})_S + \vec\Omega\times\vec r\right)$$
$$= \ddot{\vec r} + \dot{\vec\Omega}\times\vec r + \vec\Omega\times\dot{\vec r} + \vec\Omega\times\dot{\vec r} + \vec\Omega\times(\vec\Omega\times\vec r)$$
$$= \ddot{\vec r} + 2\vec\Omega\times\dot{\vec r} + \vec\Omega\times(\vec\Omega\times\vec r) + \dot{\vec\Omega}\times\vec r$$
$$\Rightarrow m\ddot{\vec r} = \vec F + 2m\dot{\vec r}\times\vec\Omega + m(\vec\Omega\times\vec r)\times\vec\Omega + m\vec r\times\dot{\vec\Omega}$$

9.14



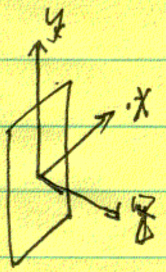
In the frame of rotating water, the water feels three forces: Gravity G, inertial force F_{CR} and N. Because $\vec N$ is along the direction perpendicular to tangential surface, we have:



$$y' = tg\theta = \frac{|F_{CR}|}{|G|} = \frac{m\omega^2 x}{mg} = \frac{\omega^2}{g}x$$
$$\Rightarrow dy = \frac{\omega^2}{g}x\,dx$$
$$\Rightarrow y = \frac{\omega^2}{2g}x^2 + \text{Constant}$$

where constant is determined by the choice of origin.

9.18



Because the particle is constrained in the vertical plane, so in the frame of rotating plane, we are only interested in the force parallel to xy plane.

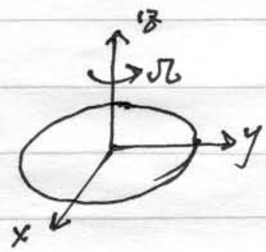
$$\Rightarrow \begin{cases} m\ddot{\vec y} = -mg\,\hat y \\ m\ddot{\vec x} = m(\vec\Omega\times\vec x)\times\vec\Omega \end{cases} \Rightarrow \begin{cases} \ddot y = -g \\ \ddot x = \Omega^2 x \end{cases}$$

where the normal force $\vec N$ and Corliolis Force are not considered for they are in $\hat{z}$ direction.

$$\begin{cases} \ddot{y} = -g \\ \ddot{x} = \Omega^2 x \end{cases} \Rightarrow \begin{cases} y = y_0 + v_{y_0} t - \frac{1}{2} g t^2 \\ x = A e^{\Omega t} + B e^{-\Omega t} \end{cases}$$

Clearly, the x and y direction motion are spearatable. In y direction, it is like the particle with initial velocity moves in the force of gravity. In x direction, if $A \neq 0$, the particle will always move away from the axis as time is large enough. However, if $A = 0$, $x = Be^{-\Omega t}$, we will find the particle moves towards the ~~origin~~ y axis and as $t \to \infty$, it will arrive.

9.20



(a) The general form of the force in the frame of rotating is:

$$m\ddot{\vec{r}} = F_{cf} + F_{cor} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} + 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\Rightarrow \begin{cases} m\ddot{x} = m\Omega^2 x + 2m\Omega\dot{y} \\ m\ddot{y} = m\Omega^2 y - 2m\Omega\dot{x} \end{cases} \quad -(20.1)$$

(b) suppose $\eta = x + iy$, we get from (20.1) that:

$$\ddot{\eta} = \Omega^2 \eta + 2\Omega \cdot (\dot{y} - i\dot{x}) = \Omega^2 \eta - 2i\Omega\dot{\eta} \quad -(20.2)$$

The general solution of this form Second order ODE is $e^{\alpha t}$, put into (20.2):

$$\alpha^2 + 2i\Omega\alpha - \Omega^2 = 0 \Rightarrow \alpha = -i\Omega$$

So the solution $\alpha = -i\Omega$ is the degenerate solution of (20.2), anothe independent solution is $e^{\alpha t} \cdot t$, therefore, we could get the general form of solution constructed by those two independent solutions:

$$\eta = e^{-i\Omega t}(C_1 + C_2 t) \quad -(20.3)$$

(c) $$\begin{cases} \vec{r}_0 = (x_0, 0) \\ \vec{v}_0 = (v_{x_0}, v_{y_0}) \end{cases} \Rightarrow \begin{cases} \eta_0 = x_0 \\ \dot{\eta}_0 = v_{x_0} + i v_{y_0} \end{cases} \quad -(20.4)$$

Combine (20.3) & (20.4), we get:

$$\eta = e^{-i\Omega t}[x_0 + v_{x_0} t + i(v_{y_0} + \Omega x_0)t] = (\cos\Omega t - i\sin\Omega t)[x_0 + v_{x_0} t + i(v_{y_0} + \Omega x_0)t]$$

$$= x + iy$$

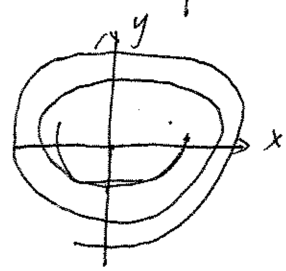
$$\Rightarrow \begin{cases} x = \cos\Omega t\,(x_0 + v_{x0} t) + \sin\Omega t\,(v_{y0} + \Omega x_0)\,t \\ y = \cos\Omega t\,(v_{y0} + \Omega x_0)\,t - \sin\Omega t\,(x_0 + v_{x0} t) \end{cases}$$

(d) If v_{y0}, Ω, x_0 and v_{x0} are not zero, we could have that, when $t \to \infty$:

$$\begin{cases} x \doteq v_{x0}\cdot t\cdot\cos\Omega t + (v_{y0} + \Omega x_0)\,t\cdot\sin\Omega t = t\,[v_{x0}\cos\Omega t + (v_{y0} + \Omega x_0)\cdot\sin\Omega t] \\ y \doteq (v_{y0} + \Omega x_0)\,t\cdot\cos\Omega t - v_{x0}\cdot t\cdot\sin\Omega t = t\,[(v_{y0} + \Omega x_0)\cos\Omega t - v_{x0}\cdot\sin\Omega t] \end{cases}$$

$$\Rightarrow \begin{cases} x = t\cdot C\cdot\cos(\Omega t - \delta) \\ y = -t\cdot C\cdot\sin(\Omega t - \delta) \end{cases} \quad \text{where, } \cos\delta = \frac{v_{x0}}{\sqrt{v_{x0}^2 + (v_{y0} + \Omega x_0)^2}},\ \sin\delta = \frac{v_{y0} + \Omega x_0}{\sqrt{v_{x0}^2 + (v_{y0} + \Omega x_0)^2}}, \quad C = \sqrt{v_{x0}^2 + (v_{y0} + \Omega x_0)^2}$$

Clearly, the motion of $t \to \infty$ is a spiral motion. the distance $r = tC$ is increased propotional to the time. The picture of the motion as t is large looks like:



7.22. In the inertial frame, we have:

$$m\left(\frac{d^2\vec r}{dt^2}\right)_{S_0} = -\frac{qQ}{r^2 4\pi\varepsilon_0}\cdot\hat r - q\left(\frac{d\vec r}{dt}\right)_{S_0}\times\vec B \qquad -(22.1)$$

Change (22.1) to the expression in the rotating frame with angular velocity Ω:

$$m\ddot{\vec r} + 2m\vec\Omega\times\dot{\vec r} + m\vec\Omega\times(\vec\Omega\times\vec r) = -\frac{qQ}{r^2 4\pi\varepsilon_0}\cdot\hat r - q(\dot{\vec r} + \vec\Omega\times\vec r)\times\vec B \qquad -(22.1)$$

where we use the conclusion in the problem (9.10).

From (22.1), we get the condition to cancle the term $\dot{\vec r}$:

$$2m\vec\Omega = +q\vec B \Rightarrow \vec\Omega = +\frac{q\vec B}{2m} \qquad -(22.2)$$

Input (22.2) into (22.1), we have:

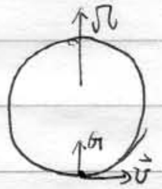
$$m\ddot{\vec r} = -\frac{qQ}{r^2 4\pi\varepsilon_0}\hat r + \frac{q^2}{4m}(\vec B\times\vec r)\times\vec B - \frac{q^2}{2m}(\vec B\times\vec r)\times\vec B$$

$$\Rightarrow m\ddot{\vec r} = -\frac{qQ}{4\pi\varepsilon_0 r^2}\hat r - \frac{q^2}{4m}(\vec B\times\vec r)\times\vec B \qquad -(22.3)$$

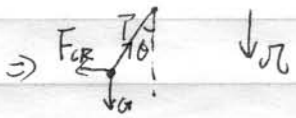
If $B \ll 1$, we have $m\ddot r = -\frac{qQ}{4\pi\varepsilon_0 r^2}$, which is the Kepler problem. The orbit is ellipse or hyperbola, which depends on the initial energy of this charge.

And in the original nonrotating frame, we could find the particle also rotate around an axis with angular velocity $|\vec{\omega}| = \frac{qB}{2m} \ll 1$. So the ellipse orbit precess in the original frame, but the frequency is small.

9.25



①



②

From Pic ①, we could find

$F_{cf} = m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = 0$

So the only inertial force is $F_{cR} = 2m\dot{\vec{r}} \times \vec{\Omega}$

And the angle θ in pic ② could be got by:

$$tg\,\theta = \frac{|F_{cR}|}{mg} \Rightarrow \theta = \mathrm{Arctan}\left[\frac{|F_{cR}|}{mg}\right] = \mathrm{Arctan}\left[\frac{2\dot{\vec{r}}\Omega}{g}\right]$$

Input the corresponding data, we get:

$$\theta = \mathrm{Arctan}\left[\frac{2\times 150\times 7.3\times 10^{-5}}{9.8}\right] \doteq 0.0023\ \mathrm{rad} = 0.13^\circ$$

And the deviation from original vertical line is in the left when seeing along the direction of motion.

9.34. Let's use $m g_{0(r)}$ to denote the gravity.

The equation of motion in the rotating frame is:

$$m\ddot{\vec{r}} = -m g_{0(\vec{R}_r)} \hat{R}_r + 2m\dot{\vec{r}} \times \vec{\Omega} + m[\vec{\Omega} \times \vec{R}_r] \times \vec{\Omega} + \vec{N} \quad - (34.1)$$

where $\vec{R}_r = \vec{R} + \vec{r}$, $\vec{R}$ is the vector from center of earth to point P, and $\vec{r}$ is vector from P to any point in the horizontal surface.

$g_{0(\vec{R}_r)}\hat{R}_r = GM \cdot \frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|^3} \doteq GM \cdot \frac{\vec{R}+\vec{r}}{R^3}$, where we use $r \ll R$ and only keep the $\vec{r}$ term.

$$\Rightarrow g_{0(\vec{R}_r)}\hat{R}_r = g_{0(\vec{R})}\hat{R} + g_{0(R)} \cdot \frac{\vec{r}}{R}, \quad g_{0(R)} = \frac{GM}{R^2}$$

And $(\vec{\Omega} \times \vec{R}_r) \times \vec{\Omega} = [\vec{\Omega} \times (\vec{R} + \vec{r})] \times \vec{\Omega} = (\vec{\Omega} \times \vec{R}) \times \vec{\Omega} + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$

$$\Rightarrow -m g_{0(R_r)}\hat{R}_r + m(\vec{\Omega} \times \vec{R}_r) \times \vec{\Omega} = -m g_{p(R)}\hat{R}_p + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} - m g_{0(R)} \frac{\vec{r}}{R} \quad - (34.2)$$

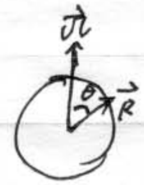
where $-g_{p(R)}\hat{R}_p$ is the effective gravity accelaration. And because

$$\frac{|(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}|}{|g_{0(R)} \cdot \frac{\vec{r}}{R}|} < \frac{\Omega^2 R}{g_0} \sim 10^{-3}$$, so we could neglect $(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$ in (34.2).

Therefore, (34.1) and (34.2) gives us:

$$\ddot{\vec{r}} = \frac{\vec{N}}{m} - g_{p(R)}\hat{R}_p + 2m\dot{\vec{r}}\times\vec{\Omega} - g_{o(R)}\frac{\vec{r}}{R}$$

where $\frac{\vec{N}}{m}$ and $g_{p(R)}\hat{R}_p$ cancles.



$$\ddot{\vec{r}} = -g_{o(R)}\frac{\vec{r}}{R} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

$$\Rightarrow \begin{cases} \ddot{x} = -gx/R + 2\dot{y}\Omega\cos\theta \\ \ddot{y} = -gy/R - 2\dot{x}\Omega\cos\theta \end{cases}$$

suppose $\eta = x+iy$, we have:

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = 0 \qquad - \quad (34.3)$$

where we use $\omega_0 = \sqrt{\frac{g}{R}}$, $\Omega_z = \Omega\cdot\cos\theta$

Input data, we have $\omega_0 \doteq 1.24\times10^{-3}$ rad/s, $\Omega_z = \Omega\cos\theta \doteq 7.3\times10^{-5}\cos\theta$ rad/s

so $\Omega_z \ll \omega_0$.

Choose the general form of solution $e^{\alpha t}$

$$\Rightarrow \alpha^2 + 2i\alpha\Omega_z + \omega_0^2 = 0 \quad \Rightarrow \alpha = -i\Omega_z \pm i\sqrt{\Omega_z^2+\omega_0^2} \doteq -i(\Omega_z + \omega_0)$$

$$\Rightarrow \eta = e^{-i\Omega_z t}(Ae^{i\omega_0 t} + Be^{-i\omega_0 t})$$

The frequency of oscillations is $\omega_0 = \sqrt{\frac{g}{R}} \doteq 1.24\times10^{-3}$ rad/s

the frequency of precession is $\Omega_z = \Omega\cos\theta \doteq 7.3\times10^{-5}\cos\theta$ rad/s

$$|F_{restore}| = |mgr/R| = \frac{mg}{R}\cdot|r| = m\omega_0^2|r|$$

$$|F_{CR}| = 2m|\dot{\vec{r}}\times\vec{\Omega}| \sim 2m\omega_0|r|\cdot\Omega$$

$$|F_{neg}| = m|(\vec{\Omega}\times\vec{r})\times\vec{\Omega}| \sim m\Omega^2|r|$$

$$\Rightarrow \begin{cases} \frac{|F_{restore}|}{|F_{CR}|} \sim \frac{\omega_0}{\Omega} \\ \frac{|F_{restore}|}{|F_{neg}|} \sim \left(\frac{\omega_0}{\Omega}\right)^2 \end{cases}$$