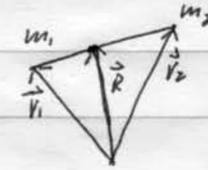
Solutions of PHYS 410 Homework #5

8.6. In CM frame, we have:



$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = 0, \quad \vec{P} = \vec{p}_1 + \vec{p}_2 = 0$$

$$\Rightarrow \vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1, \quad \vec{p}_2 = -\vec{p}_1$$

$$\Rightarrow \vec{L} = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2 = \left(\frac{m_1}{m_2}+1\right)\vec{r}_1\times\vec{p}_1 = \frac{M}{m_2}\vec{L}_1$$

$$= \left(\frac{m_2}{m_1}+1\right)\vec{r}_2\times\vec{p}_2 = \frac{M}{m_1}\vec{L}_2$$

$$\Rightarrow \vec{L}_1 = \frac{m_2}{M}\vec{L}, \quad \vec{L}_2 = \frac{m_1}{M}\vec{L}$$

8.11. From the equation $\mu\ddot{\vec{r}} = -\nabla U(\vec{r}) = -k\vec{r}$,

we have $\begin{cases} \mu\ddot{x} = -kx \\ \mu\ddot{y} = -ky \end{cases}$

So we have the general form of (x,y):

$\begin{cases} x = A\cos\omega t + B\sin\omega t \\ y = C\cos\omega t + D\sin\omega t \end{cases}$, where A, B, C, D are all constants

$$\Rightarrow \begin{cases} \sin\omega t = (Cx - Ay)/(BC-AD) \\ \cos\omega t = (By - Dx)/(BC-AD) \end{cases} \Rightarrow (Cx-Ay)^2 + (By-Dx)^2 = (BC-AD)^2$$

$$\Rightarrow (C^2+D^2)x^2 - 2(AC+BD)xy + (A^2+B^2)y^2 = (BC-AD)^2 \quad \text{— (11.1)}$$

If $BC-AD \neq 0$, we have $(C^2+D^2)(A^2+B^2) - [-(AC+BD)]^2 = (BC-AD)^2 > 0$

So equation (11.1) gives us an ellipse equation.

If $BC-AD=0$, (11.1) gives us:

$$\left(\sqrt{C^2+D^2}\,x - \sqrt{A^2+B^2}\,y\right)^2 = 0 \Rightarrow y = \frac{\sqrt{C^2+D^2}}{\sqrt{A^2+B^2}}x$$

Which is a line, a special case of ellipse.

8.12. (a) $\begin{cases} \mu\ddot{r} = -\nabla U_{eff} \\ U_{eff} = -\dfrac{Gm_1m_2}{r} + \dfrac{l^2}{2\mu r^2} \end{cases} \Rightarrow$ circular orbit corresponds to $\left.\dfrac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = 0$

$$\Rightarrow \frac{Gm_1m_2}{r_0^2} - \frac{l^2}{\mu r_0^3} \Rightarrow l^2 = \frac{Gm_1m_2}{r_0^2}\cdot\mu r_0^3 = Gm_1m_2\cdot\mu r_0$$

(b) For small radial perturbation, we could rewrite $\delta r = r_0 + \Delta r$,

So $\mu \ddot{r} = -\nabla V_{eff}$ changes to:

$$\mu \ddot{\Delta r} = - \frac{\partial^2 V_{eff}}{\partial r^2}\Big|_{r=r_0} \cdot \Delta r \ ;$$ which is of the form of small oscillation

$\Rightarrow$ The angular frequency of this small oscillation is

$$\omega^2 = \frac{1}{\mu} \frac{\partial^2 V_{eff}}{\partial r^2}\Big|_{r=r_0} = \frac{1}{\mu}\left(-\frac{2Gm_1m_2}{r_0^3} + \frac{3l^2}{\mu r_0^4}\right) = \frac{1}{\mu} \cdot \frac{l^2}{\mu r_0^4} = \left(\frac{l}{\mu r_0^2}\right)^2 = \omega_0^2$$

$\Rightarrow$ $\omega = \omega_0 = \frac{l}{\mu r_0}$, which is the angular frequency of orbital period.

8.15. (8.54) gives us exactly, for ellipse orbit, we have:

$$\tau^2 = 4\pi^2 \frac{a^3 \mu}{\gamma}$$

and $\gamma = G m_1 m_2 = G \cdot \mu \cdot M$

$$\Rightarrow \tau^2 = \frac{4\pi^2}{GM} a^3 = \frac{4\pi^2}{G(m_1+m_2)} a^3$$

if $m_1 = 2\times10^{27}$ kg, $m_2 = 2\times10^{30}$

we have $\tau_1^2 = \frac{4\pi a^3}{G} \cdot \frac{1}{m_1+m_2}$

$$\tau_2^2 = \frac{4\pi a^3}{G} \cdot \frac{1}{m_2} = \tau_1^2 \cdot \frac{m_1+m_2}{m_2}$$

$$\Rightarrow \tau_2 = \tau_1 \cdot \left(\frac{m_1+m_2}{m_2}\right)^{1/2} \doteq \tau_1 \left(1+\frac{m_1}{2m_2}\right)$$

$$\Rightarrow \frac{\Delta\tau}{\tau_1} = \frac{m_1}{2m_2} = \frac{1}{2\times10^3} = 5\times10^{-4}$$

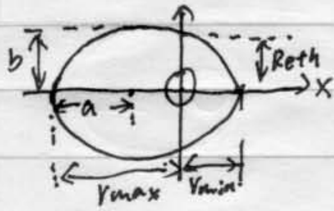
8.19. From (8.50) on the textbook, we have:

$$r_{min} = \frac{c}{1+\epsilon} , \quad r_{max} = \frac{c}{1-\epsilon}$$

$$\Rightarrow \frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon} \Rightarrow \epsilon = \frac{r_{max}-r_{min}}{r_{max}+r_{min}}$$

$r_{max} = R_e + 3000 = 9.4\times10^3$ km, $\quad r_{min} = R_e + 300 = 6.7\times10^3$ km

$\Rightarrow \epsilon \doteq 0.17$



$$R_{eth} = \frac{b^2}{a} = a(1-\epsilon^2) = \frac{r_{min}+r_{max}}{2}\cdot(1-\epsilon^2)$$

$$\Rightarrow h = \frac{9.4\times10^3+6.7\times10^3}{2}(1-0.17^2) - 6.4\times10^3$$

$$\doteq 1.4\times10^3\ \text{Km}$$

8.24 From equation (8.41) on the textbook, we have:

$$u''(\phi) = -u(\phi) - \frac{\mu}{l^2u^2(\phi)}\cdot F$$

$$= -u(\phi) - \frac{\mu}{l^2u^2}(-\cancel{\beta}ku^2 + \lambda u^3)$$

$$= -(1+\frac{\mu\lambda}{l^2})u(\phi) + \frac{\mu k}{l^2}$$

For $l^2 < -\mu\lambda$, we have $-(1+\frac{\mu\lambda}{l^2}) > 0$

Denote $\alpha^2 = -(1+\frac{\mu\lambda}{l^2})$, $\beta = \frac{\mu k}{l^2\alpha^2} > 0$

we have $u''(\phi) = \alpha^2(u(\phi)+\beta)$

$$\Rightarrow u(\phi)+\beta = Ae^{\alpha\phi} + Be^{-\alpha\phi}$$

$$\Rightarrow u(\phi) = -\beta + Ae^{\alpha\phi} + Be^{-\alpha\phi}$$

Considering ① $\phi(t)$ always increases with time

② $u(\phi)$ always can't be negative

we have

$$\begin{cases} A > 0 \\ Ae^{\alpha\phi_0} + Be^{-\alpha\phi_0} - \beta \geq 0 \qquad (\phi_0 = \phi(t=0)) \end{cases}$$

and $u(\phi)$ increase with $\phi(t)$, that is, increase with time.

So, the orbit could only has one infinity point ~~when~~ at begining, at could not has infinity point when $t \neq 0$. That is $u(\phi)$ could ~~only~~ be zero only at begining, otherw $u(\phi)$ could not be zero. And as time goes on, the radial distance will decreases, as time $t \to \infty$, we get $u \to \infty$ and $r \to 0$.

8.30 From equation (8.49), we have:

$$r = \frac{c}{1+\epsilon\cos\phi} \Rightarrow r + \epsilon r\cos\phi = c \Rightarrow r + x\epsilon = c \Rightarrow r = c - \epsilon x$$

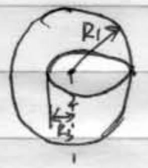
$$\Rightarrow r^2 = x^2+y^2 = (c-\epsilon x)^2 = c^2 - 2c\epsilon x + \epsilon^2x^2$$

① $\epsilon = 1 \Rightarrow y^2 = c^2 - 2cx$

② $\epsilon > 1 \Rightarrow (\sqrt{\epsilon^2-1}\,x - \frac{c\epsilon}{\sqrt{\epsilon^2-1}})^2 - y^2 = c^2[(\frac{\epsilon}{\sqrt{\epsilon^2-1}})^2 - 1] = c^2\cdot\frac{1}{\epsilon^2-1}$

$$\Rightarrow \frac{\left(x-\frac{c\epsilon}{\epsilon^2-1}\right)^2}{\left(\frac{c}{\epsilon^2-1}\right)^2}-\frac{y^2}{\left(\frac{c}{\sqrt{\epsilon^2-1}}\right)^2}=1$$

8.35



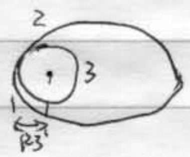
$$\begin{cases}\frac{c_1}{1+\epsilon_1}=\frac{c_2}{1+\epsilon_2}\\ c_2=\lambda^2 c_1\end{cases}\Rightarrow \epsilon_2=\lambda^2(1+\epsilon_1)-1$$

Because $\epsilon_1=0 \Rightarrow \epsilon_2=\lambda^2-1$

$$R_3=\frac{c_2}{1-\epsilon_2}=\frac{\lambda^2 R_1}{1-(\lambda^2-1)}=\frac{\lambda^2 R_1}{2-\lambda^2}$$

$$\Rightarrow \lambda^2=\frac{2R_3}{R_1+R_3}=\frac{\frac{1}{2}R}{\frac{5}{4}R}=\frac{2}{5}$$

$$\Rightarrow \lambda=\sqrt{\frac{2}{5}}\doteq 0.63$$



$$\begin{cases}\frac{c_2}{1-\epsilon_2}=\frac{c_3}{1+\epsilon_3}=c_3\\ c_3=\lambda'^2 c_2\end{cases}\Rightarrow \lambda'^2=\frac{1}{1-\epsilon_2}=\frac{1}{2-\lambda^2}=\frac{1}{2-\frac{2R_3}{R_1+R_3}}=\frac{R_1+R_3}{2R_1}$$

$$\Rightarrow \lambda'=\sqrt{\frac{R_1+R_3}{2R_1}}=\sqrt{\frac{5}{8}}\doteq 0.79$$

$$v_{2(apo)}R_3=v_{2(per)}R_1$$

$$\Rightarrow v_3=\lambda' v_{2(apo)}=\lambda'\frac{R_1}{R_3}\cdot v_{2(per)}=\lambda'\frac{R_1}{R_3}\cdot\lambda\cdot v_1$$

$$=\sqrt{\frac{2R_3}{R_1+R_3}}\cdot\frac{R_1}{R_3}\cdot\sqrt{\frac{R_1+R_3}{2R_1}}\cdot v_1$$

$$=\sqrt{\frac{R_1}{R_3}}\cdot v_1=2v_1$$