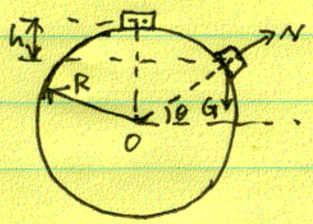
PHYS 410 Homework # 4 Solutions

4.2 (a) $W = \int_0^P \vec{F}\cdot d\vec{r} = \int_0^P (F_x dx + F_y dy) = \int_0^1 x^2 dx + \int_0^1 2y\,dy = \frac{1}{3} + 1 = \frac{4}{3}$

(b) $W = \int_0^P (F_x dx + F_y dy) = \int_0^1 x^2 dx + \int_0^1 2x\cdot x^2 dx^2 = \frac{1}{3} + \frac{4}{5} = \frac{17}{15}$

(c) $W = \int_0^P (F_x dx + F_y dy) = \int_0^1 (t^3)^2 dt^3 + \int_0^1 2\cdot t^3\cdot t^2 dt^2 = \frac{1}{3} + \frac{4}{7} = \frac{19}{21}$

4.8



Choose the gravity potential as zero at O.

So, from the conversation of Energy and denotion of pic, we have

$mgR - mgh + \frac{1}{2}mv^2 = mgR$

$\Rightarrow v^2 = 2gh$ — ①

And, from the central motion of the mass, we have:

$m\frac{v^2}{R} = G\cdot\sin\theta - N = mg\sin\theta - N$

$\sin\theta = \frac{R-h}{R}$, and $N=0$ when the mass is to leave:

$v^2 = gR\cdot\frac{R-h}{R}$ — ②

From ① & ②, we have $h = \frac{R}{3}$.

4.10. (a) $\frac{\partial f}{\partial x} = 2ax + by$, $\frac{\partial f}{\partial y} = bx + 2cy$, $\frac{\partial f}{\partial z} = 0$

(b) $\frac{\partial g}{\partial x} = a\cos(axyz^2)\cdot yz^2 = ayz^2\cdot\cos(axyz^2)$

$\frac{\partial g}{\partial y} = axz^2\cos(axyz^2)$, $\frac{\partial g}{\partial z} = 2axyz\cos(axyz^2)$

(c) $\frac{\partial h}{\partial x} = \frac{ay}{z^2}e^{\frac{xy}{z^2}}$, $\frac{\partial h}{\partial y} = \frac{ax}{z^2}e^{\frac{xy}{z^2}}$, $\frac{\partial h}{\partial z} = -\frac{2axy}{z^3}e^{\frac{xy}{z^2}}$

4.11 (a) $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2ay + 2bz$, $\frac{\partial f}{\partial z} = 2by + 2cz$

(b) $\frac{\partial g}{\partial x} = -ay^2z^3\sin(axy^2z^3)$, $\frac{\partial g}{\partial y} = -2axyz^3\sin(axy^2z^3)$, $\frac{\partial g}{\partial z} = -3axy^2z^2\sin(axy^2z^3)$

(c) $\frac{\partial h}{\partial x} = a \cdot \frac{\partial r}{\partial x} = a \cdot \frac{x}{r}$, $\frac{\partial h}{\partial y} = a\frac{y}{r}$, $\frac{\partial h}{\partial z} = a \cdot \frac{z}{r}$

4.20 (a) $\vec{F} = k\vec{r} = \nabla(\frac{1}{2}kr^2) \Rightarrow \nabla \times \vec{F} = \nabla \times \nabla(\frac{1}{2}kr^2) = 0$

(b) $\vec{F} = (Ax, By^2, Cz^3) \Rightarrow \vec{F} = \nabla(\frac{1}{2}Ax^2 + \frac{1}{3}By^3 + \frac{1}{4}Cz^4) \Rightarrow \nabla \times \vec{F} = \nabla \times V(x,y,z) = 0$

(c) $\vec{F} = (Ay^2, Bx, Cz) \Rightarrow \nabla \times \vec{F} = (B - 2Ay)\vec{e}_z$

4.28 (a) $\vec{E} = E(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$\Rightarrow \dot{x} = (\frac{2E}{m} - \frac{k}{m}x^2)^{1/2}$

(b) when $x = A$, we have $\dot{x} = 0 \Rightarrow E = \frac{1}{2}kA^2$

$\Rightarrow \dot{x} = \sqrt{\frac{k}{m}}(A^2 - x^2)^{1/2}$

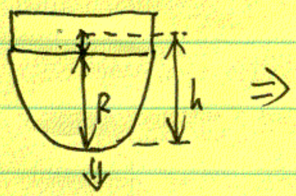
For the first 1/4 cycle, we have:

$t = \int_0^x \frac{dx}{\dot{x}} = \sqrt{\frac{m}{k}} \int_0^x \frac{dx}{(A^2 - x^2)^{1/2}} = \sqrt{\frac{m}{k}} \sin^{-1}(\frac{x}{A})$

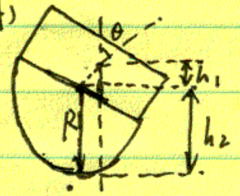
(c) $t = \sqrt{\frac{m}{k}} \sin^{-1}(\frac{x}{A}) \Rightarrow x = A \sin\sqrt{\frac{k}{m}}t$ which is right in all time

$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$

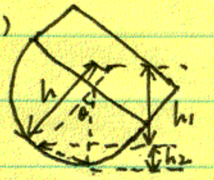
4.30



(A)



(B)



Two approaches:

Approach A: (a) See pic A.

We have $h(\theta) = h_1 + h_2$

$= (h - R) \cdot \cos\theta + R$

$\Rightarrow U = mgh(\theta) = mg[R + (h - R)\cos\theta]$

(b) $\frac{\partial U}{\partial \theta} = -mg(h - R)\sin\theta = 0 \Rightarrow \theta = 0$

$\frac{\partial^2 U}{\partial \theta^2}\Big|_{\theta=0} = mg(R - h) \Rightarrow \begin{cases} R > h \Rightarrow \frac{\partial^2 U}{\partial \theta^2}\Big|_{\theta=0} > 0, & \text{equilibrium is stable} \\ R < h \Rightarrow \frac{\partial^2 U}{\partial \theta^2}\Big|_{\theta=0} < 0, & \text{equilibrium is unstable.} \end{cases}$

Approach B: (a) see pic B, suppose θ is small.

we have $h(\theta) = h_1 + h_2 = h\cdot\cos\theta + R\theta\cdot\cos\frac{\pi-\theta}{2} = h\cos\theta + R\theta\sin\frac{\theta}{2}$

$\Rightarrow U = mgh(\theta) = mg[h\cos\theta + R\theta\sin\frac{\theta}{2}]$

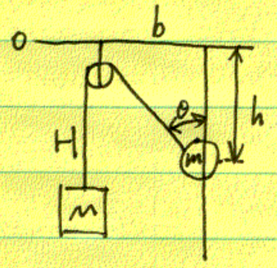
(b) $\frac{\partial U}{\partial\theta} = mg[-h\sin\theta + R\sin\frac{\theta}{2} + \frac{1}{2}R\theta\cos\frac{\theta}{2}] = 0 \Rightarrow \theta = 0$

$\frac{\partial^2 U}{\partial\theta^2}|(\theta=0) = mg[-h\cos\theta + \frac{1}{2}R\cos\frac{\theta}{2} + \frac{1}{2}R\cos\frac{\theta}{2} - \frac{1}{4}R\theta\sin\frac{\theta}{2}]\Big|_{\theta=0}$

$= mg(R-h)$

$\Rightarrow \begin{cases} R>h \Rightarrow \frac{\partial^2 U}{\partial\theta^2}|_{\theta=0} > 0, & \text{equilibrium is stable} \\ R<h \Rightarrow \frac{\partial^2 U}{\partial\theta^2}|_{\theta=0} < 0, & \text{equilibrium is unstable} \end{cases}$

4.36



(a) Choose the zero potential at the top celling.
And suppose the length of the line is l.

So $h = b/tg\theta$, $H = l - b/\sin\theta$

$\Rightarrow U(\theta) = -mgh - MgH$

$= -mgb/tg\theta - Mgl + Mgb/\sin\theta$

$= -Mgl + gb\left(\frac{M}{\sin\theta} - \frac{m}{tg\theta}\right)$

(b) $\frac{\partial U}{\partial\theta} = gb\cdot\left[-\frac{M}{\sin^2\theta}\cdot\cos\theta + \frac{m}{\sin^2\theta}\right] = 0$

$\Rightarrow m = M\cos\theta$.

If $m<M$, $\cos\theta = \frac{m}{M}$ is the equilibrium point; else, no equilibrium point.

For this equilibrium point, we have:

$\frac{\partial^2 U}{\partial\theta^2}|(\theta=\arccos\frac{m}{M}) = gb\cdot\left[-2\cdot\frac{(m-M\cos\theta)\cos\theta}{\sin^3\theta} + \frac{M\sin\theta}{\sin^2\theta}\right]\Big|_{\cos\theta=\frac{m}{M}}$

$= gb\cdot\frac{M+\frac{m^2}{M}-2\frac{m^2}{M}}{\sin^3\theta_0} > 0$

$\Rightarrow$ This equilibrium is stable.

4.43 (a) In cartesian coordinates, we have:

$(\nabla\times\vec{F}(\vec{r}))_x = \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y = \frac{\partial}{\partial y}\left(f(r)\cdot\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(f(r)\frac{y}{r}\right)$

Because $\frac{\partial}{\partial y}f(r) = f'(r)\frac{\partial r}{\partial y} = f'(r)\frac{y}{r}$, $\frac{\partial}{\partial z}f(r) = f'(r)\frac{z}{r}$

$$\Rightarrow \frac{\partial}{\partial y}\left(f(r)\cdot\frac{z}{r}\right) = f'(r)\cdot\frac{y}{r}\cdot\frac{z}{r} + \left[\frac{\partial}{\partial y}\left(\frac{z}{r}\right)\right]f(r) = f'(r)\cdot\frac{yz}{r^2} - \frac{yz}{r^3}f(r)$$

$$\frac{\partial}{\partial z}\left(f(r)\cdot\frac{y}{r}\right) = f'(r)\frac{yz}{r^3} - \frac{yz}{r^3}f(r)$$

$$\Rightarrow (\nabla\times\vec{F}(\vec{r}))_x = 0$$

Similarly, $(\nabla\times\vec{F}(\vec{r}))_y = (\nabla\times\vec{F}(\vec{r}))_z = 0$

$$\Rightarrow \nabla\times\vec{F}(\vec{r}) = 0$$

(b) Because $F(r)$ has no other components than $\hat{r}$.

So $\nabla\times\vec{F}(r) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times(F(r)\hat{r}) = 0$

4.48. In this process, the momentum of the system of m_1 and m_2 is conserved:

$$m_1 v_1 = (m_1+m_2)v_f$$

$$\Rightarrow v_f = \frac{m_1}{m_1+m_2}v_1$$

$$\Rightarrow E_f = \frac{1}{2}(m_1+m_2)\cdot v_f^2 = \frac{1}{2}\cdot\frac{m_1^2}{m_1+m_2}v_1^2$$

$$\Rightarrow E_{loss}/E_i = \left(\frac{1}{2}m_1 v_1^2 - \frac{1}{2}\frac{m_1^2}{m_1+m_2}v_1^2\right)\Big/\frac{1}{2}m_1 v_1^2 = 1 - \frac{m_1}{m_1+m_2} = \frac{m_2}{m_1+m_2}$$

when $m_1 \gg m_2$, we have $E_{loss}/E_i \to 0$

$m_1 \ll m_2$, we have $E_{loss}/E_i \to 1$