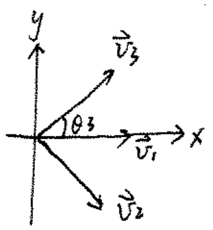
Solutions of Homework #3

.3



According to the conversation of Momentum, we have:

$$m\vec{v}_0 = \frac{m}{3}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$$

$$\Rightarrow \begin{cases} X \text{ direction: } & m v_0 = \frac{m}{3}(v_0 + v_3\cdot\cos\theta_3 + v_2\cdot\cos\theta_2) & -(3.1) \\ y \text{ direction: } & 0 = \frac{m}{3}(v_3\sin\theta_3 - v_2\sin\theta_2) & -(3.2) \end{cases}$$

And, $$\begin{cases} v_3 = v_2 & -(3.3) \\ \theta_2 + \theta_3 = \frac{\pi}{2} & -(3.4) \end{cases}$$

Combing (3.1) ~ (3.4), we get: $\theta_2 = \theta_3 = \frac{\pi}{4}$, $v_2 = v_3 = v_0$

3.8. (a) Because $m\dot{v} = -\dot{m}v_{ex} - mg = 0$

We have $\dot{m}v_{ex} = -mg \Rightarrow dt = -\frac{v_{ex}}{g}\cdot\frac{dm}{m} \Rightarrow t = \frac{v_{ex}}{g}\ln\left(\frac{m_0}{m_0-\lambda m_0}\right) = \frac{v_{ex}}{g}\ln\left(\frac{1}{1-\lambda}\right)$

(b) $v_{ex} = 3000\,m/s$, $\lambda = 0.9$, we have $\Delta t \doteq 32.3s$

.13. $m\dot{v} = -\dot{m}v_{ex} - mg \Rightarrow \dot{v} = -\frac{\dot{m}}{m}v_{ex} - g \Rightarrow v(t) = -\int_0^t \frac{-k}{m_0-kt}v_{ex}\,dt - gt = v_{ex}\ln\frac{m_0}{m(t)} - gt$

$$y(t) = \int_0^t v(t)\,dt = -\frac{1}{2}gt^2 + \int_0^t v_{ex}\ln\frac{m_0}{m(t)}\,dt = -\frac{1}{2}gt^2 + \left(-\frac{v_{ex}}{k}\right)\int_{m_0}^{m(t)}\ln\frac{m_0}{m}\,dm$$

$$\int_{m_0}^{m(t)}\ln\frac{m_0}{m}\,dm = m\ln\frac{m_0}{m}\Big|_{m_0}^{m(t)} - \int_{m_0}^{m(t)} m\cdot\left(-\frac{1}{m}\right)dm = m\left(\ln\frac{m_0}{m}+1\right)\Big|_{m_0}^{m(t)} = m\ln\frac{m_0}{m} + m - m_0$$

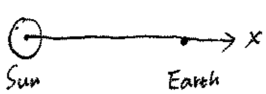
$$\Rightarrow y(t) = -\frac{1}{2}gt^2 - \frac{v_{ex}}{k}\left(m\ln\frac{m_0}{m} - kt\right) = v_{ex}t - \frac{1}{2}gt^2 - \frac{v_{ex}m}{k}\ln\frac{m_0}{m}$$

For $v_{ex} = 3\times10^3\,m/s$, $m_0 = 2\times10^6\,kg$, $m = 1\times10^6\,kg$, $t = 120s$

we have $k = \frac{1\times10^6}{120}\,kg/s$

$y \doteq 3.99\times10^4\,m$

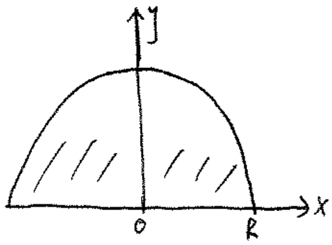
.16



$$X_{cm} = \frac{M_S\cdot X_S + M_E\cdot X_E}{M_S+M_E} \doteq \frac{M_E}{M_S}\cdot X_E = \frac{6.0\times10^{24}}{2.0\times10^{30}}\times1.5\times10^8\,km = 450\,km$$

So $X_{cm} < R_S$, which means the position of CM is very close to the center of the sun.

21



According to the symmetry, We have $X_{cm} = 0$

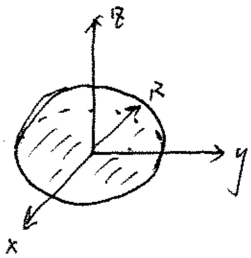
$$y_{cm} = \frac{\int y\sigma dA}{M} = \frac{\int y dA}{A} = \frac{\int_0^{\pi}\int_0^{R} r^2 \sin\theta dr d\theta}{\frac{\pi R^2}{2}} = \frac{4R}{3\pi}$$

So, the position of the CM is $(0, \frac{4R}{3\pi})$

Note: ① we could always choose x.y in the plane of the metal, which is 2D problem

② we could not directly use polar coordinates to determin cm, for $\vec{\phi}$ and $\vec{r}$ are not well-defined direction.

3.22



According to the symmetry, we could easily have:

$$X_{cm} = y_{cm} = 0$$

$$z_{cm} = \frac{\int z \rho dV}{M} = \frac{\int_0^R\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^3 \cos\theta \sin\theta d\varphi d\theta}{\frac{2}{3}\pi R^3} = \frac{3}{8}R$$

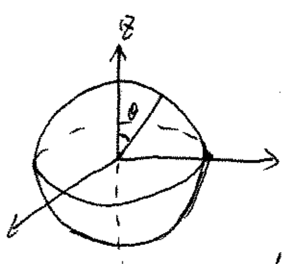
$$\vec{R}_{cm} = (0, 0, \frac{3}{8}R)$$

3.25. Because there is no torque about the origin of the circle, we have:

$$l_i = l_f \Rightarrow m\vec{r}_0 \times (\vec{w}_0 \times \vec{r}_0) = m\vec{r}\times(\vec{w}\times\vec{r})$$

$$\Rightarrow w = \frac{r_0^2}{r^2} w_0, \quad \vec{e}_w = \vec{e}_{w_0}$$

3.32.



For this problem, let's choose z axis as the axis about which the sphere is rotating. So, a point in the sphere (r, θ, φ), has a distance to z:

$$d = r\cdot|\sin\theta|$$

We have the expression of the moment of inertia:

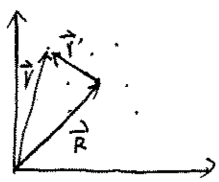
$$I = \int_V r^2 \cdot \sin^2\theta \cdot \rho dv = \frac{M}{V}\int_V r^4 dr \cdot \sin^3\theta d\theta \cdot d\varphi = \frac{M}{\frac{4}{3}\pi R^3}\cdot\frac{R^5}{5}\times\frac{4}{3}\times 2\pi = \frac{2}{5}MR^2$$

Note: Compared with 3.21, we could directly use spherical polar coordinates, that's because that the quantity calculated here has nothing to do with well-defined direction, or, it's a quantity only showing the quality of distance.

3.34. Take the CM frame, in this frame, there is no torque about the CM point. So we conclude hat the angular velocity of the rod rotating around its cm point is a constant. So, to catch to rod when it rotates exactly n rounds, we have:

$$t = n \cdot \frac{2\pi}{w_0} = \frac{2v_0}{g} \Rightarrow v_0 = \frac{n\pi g}{w_0}$$

.37 (a)



(b) $\sum m_\alpha r_\alpha' = \sum m_\alpha (\vec{r}_\alpha - \vec{R}) = \sum m_\alpha \vec{r}_\alpha - (\sum m_\alpha)\vec{R} = (\sum m_\alpha)\vec{R} - (\sum m_\alpha)\vec{R} = 0$

$\sum m_\alpha r_\alpha'$ is the position of the cm point in the frame which origin is just the cm point, so obviously $\sum m_\alpha r_\alpha' = 0$

(c) $\vec{L}_{(\text{about cm})} = \sum m_\alpha \vec{r}_\alpha' \times \dot{\vec{r}}_\alpha'$

$\Rightarrow \dot{\vec{L}}_{(\text{about cm})} = \sum m_\alpha \dot{\vec{r}}_\alpha' \times \dot{\vec{r}}_\alpha' + \sum m_\alpha \vec{r}_\alpha' \times \ddot{\vec{r}}_\alpha' = \sum m_\alpha \vec{r}_\alpha' \times (\ddot{\vec{r}}_\alpha - \ddot{\vec{R}}) = \sum m_\alpha \vec{r}_\alpha' \times \ddot{\vec{r}}_\alpha - (\sum m_\alpha \vec{r}_\alpha')\ddot{\vec{R}}$

$= \sum \vec{r}_\alpha' \times F_\alpha = \sum \vec{r}_\alpha' \times \vec{F}_{ext} + \sum_\alpha \vec{r}_\alpha' \times \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$

$\sum_\alpha \vec{r}_\alpha' \times \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} = \frac{1}{2} \sum_{\alpha \neq \beta} (\vec{r}_\alpha' \times \vec{F}_{\alpha\beta} + \vec{r}_\beta' \times \vec{F}_{\beta\alpha}) = \frac{1}{2} \sum_{\alpha \neq \beta} (\vec{r}_{\alpha\beta} \times \vec{F}_{\alpha\beta})$

For centeral internal forces, we have $\sum_\alpha \vec{r}_\alpha' \times \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} = 0$

$\Rightarrow \dot{\vec{L}}_{(\text{about cm})} = \sum \vec{r}_\alpha' \times \vec{F}_{ext} = \Gamma^{ext}_{(\text{about the cm})}$