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PHYSICS 410 — Prof. S. J. Gates

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Problem (1.)

(a.) The formal expressions for the center of mass can be found in the textbook. However, as in the presentation discussing the practice examination, we can use symmetry arguments to deduce the location is given by

$$\vec{r}_{CM} = \tfrac{1}{2}\left[\, a\,\hat{x} + b\,\hat{y} + c\,\hat{z} \,\right]$$

(b.) Find the explicit form of the moment of inertia tensor, $\mathcal{I}_{i\,j}$.

$$\mathcal{I}_{i\,j} = \frac{\mathcal{M}}{a\,b\,c}\int_0^c dz \int_0^b dy \int_0^a dx \begin{bmatrix} \left(y^2 + z^2\right) & -\,x\,y & -\,x\,z \\ -\,x\,y & \left(x^2 + z^2\right) & -\,y\,z \\ -\,z\,x & -\,z\,y & \left(x^2 + y^2\right) \end{bmatrix}$$

$$= \mathcal{M}\begin{bmatrix} \tfrac{1}{3}\,[\,b^2 + c^2\,] & -\,\frac{1}{4}\,a\,b & -\,\frac{1}{4}\,a\,c \\ -\,\frac{1}{4}\,a\,b & \tfrac{1}{3}\,[\,a^2 + c^2\,] & -\,\frac{1}{4}\,b\,c \\ -\,\frac{1}{4}\,a\,c & -\,\frac{1}{4}\,b\,c & \tfrac{1}{3}\,[\,a^2 + b^2\,] \end{bmatrix}$$

(c.) The other coordinate system is simply the one where the half the object is above the new x-y plane and half below. A direct calculation yields

$$\mathcal{I}_{i\,j} = \mathcal{M}\begin{bmatrix} \tfrac{1}{3}\,[\,b^2 + \tfrac{1}{4}\,c^2\,] & -\,\frac{1}{4}\,a\,b & 0 \\ -\,\frac{1}{4}\,a\,b & \tfrac{1}{3}\,[\,a^2 + \tfrac{1}{4}\,c^2\,] & 0 \\ 0 & 0 & \tfrac{1}{3}\,[\,a^2 + b^2\,] \end{bmatrix}$$

which is of the form,

$$\mathcal{I}_{i\,j} = \mathcal{M}\,\{\,a^2 + b^2 + c^2\,\}\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}$$

where the m's can be seen by simple comparison. This is block diagonal and one principal axis is given by

$$\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the other two principal axes are found from solving the eigenvalue problem for

$$\mathbf{M} \;=\; \mathcal{M}\,\{\,a^2 \,+\, b^2 \,+\, c^2\,\}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$$

as a simple 2×2 matrix problem. If $(w_1,\, w_2)$ is an eigenvector, then

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \;=\; \tilde{\lambda}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This leads to the requirement that

$$(\tilde{\lambda})^2 \;-\; (\,m_{11} \,+\, m_{22}\,)\,\tilde{\lambda} \;+\; \Big(m_{11}\,m_{22} \;-\; (m_{12})^2\Big) \;=\; 0$$

which implies

$$\tilde{\lambda}_{\pm} \;=\; \tfrac{1}{2}\Big[\,(\,m_{11} \,+\, m_{22}\,) \pm \sqrt{\,(\,m_{11} \,+\, m_{22}\,)^2 \,-\, 4\,(m_{11}\,m_{22} \;-\; (m_{12})^2)\,}\;\Big]$$

Two solutions of this are given by

$$w_{1\pm} = \;-\; m_{12}$$

$$w_{2\pm} = \;\tfrac{1}{2}\Big[\,(\,m_{11} \,-\, m_{22}\,) \mp \sqrt{\,(\,m_{11} \,+\, m_{22}\,)^2 \,-\, 4\,(m_{11}\,m_{22} \;-\; (m_{12})^2)\,}\;\Big]$$

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$$\widehat{e}_{+} \;=\; \begin{bmatrix} -\,m_{12} \\ \\ \frac{1}{2}\Big[\,(m_{11} - m_{22}) - \sqrt{(m_{11} + m_{22})^2 \;-\; 4\,(m_{11}m_{22} - (m_{12})^2)}\,\Big] \\ \\ 0 \end{bmatrix}$$

and

$$\widehat{e}_{-} \;=\; \begin{bmatrix} -\,m_{12} \\ \\ \frac{1}{2}\Big[\,(m_{11} - m_{22}) + \sqrt{(m_{11} + m_{22})^2 \;-\; 4\,(m_{11}m_{22} - (m_{12})^2)}\,\Big] \\ \\ 0 \end{bmatrix}$$

Problem (2.)

(a.) The equation of motion for the electron launched vertically upward takes the form

$$\frac{dv_y}{d\,t} \;=\; -\;\frac{c}{m}\,(\,v_y\,)^2$$

and we divide by $(v_y)^2$ and find

$$v_y(t) \;=\; \frac{V_0}{\left[1 \;+\; \frac{c\,V_0\,t}{m}\right]}$$

after performing an integration. A second integral can be performed to find

$$y(t) \;=\; (\tfrac{m}{c})\ln\left[1 \;+\; \frac{c\,V_0\,t}{m}\right]$$

(b.) If we assume the electron is effectively at rest when it reaches a velocity that is 10^{-3} of its initial velocity, then we must have $\left[1 \;+\; \frac{c\,V_0\,T}{m}\right] = 10^3$ for the time T when this occurs. At this time, the y-position of the electron is given by

$$y(T) \;=\; 3\,(\tfrac{m}{c})\ln 10$$

Problem (3.)

(a.) First we can find the direction of the piece of ball just prior to the collision. Since there are no forces acting on the ball from the time it is thrown until the collision with the globe, it travels in a straight line with a constant speed. The displacement vector is thus $\frac{1}{\sqrt{3}}R[\,\widehat{x}+\widehat{y}+\widehat{z}\,] \,-\, \alpha R\widehat{x}$ and the unit vector describing this direction is

$$\widehat{U} \;=\; \frac{[\,(1 \,-\, \sqrt{3}\,\alpha)\,\widehat{x}+\widehat{y}+\widehat{z}\,]}{\sqrt{2 \,+\, (1 \,-\, \sqrt{3}\,\alpha)^2}}$$

and the total momentum $\vec{P}_T$ before the collision must be

$$\vec{P}_T \;=\; m\,V\widehat{U} \;=\; \frac{m\,V\,[\,(1 \,-\, \sqrt{3}\,\alpha)\,\widehat{x}+\widehat{y}+\widehat{z}\,]}{\sqrt{2 \,+\, (1 \,-\, \sqrt{3}\,\alpha)^2}}$$

The momentum of the ball after the collision is given by

$$\vec{p}_b \;=\; m\,\vec{v}_b \;=\; \left[\,\frac{m\,V\,(\,\widehat{y} \,+\, \widehat{z}\,)}{\sqrt{(1-\alpha)^2 \,+\, 2}}\,\right] \quad .$$

. The momentum of the globe $\vec{p}_G$ must be given by

$$\vec{p}_G \;=\; \Big[\frac{m\,V\,[\,(1\,-\,\sqrt{3}\,\alpha)\,\widehat{x} + \widehat{y} + \widehat{z}]}{\sqrt{2\,+\,(1\,-\,\sqrt{3}\,\alpha)^2}}\Big] \;-\; \Big[\frac{m\,V\,(\,\widehat{y}\,+\,\widehat{z}\,)}{\sqrt{(1-\alpha)^2\,+\,2}}\Big]$$

$$=\; \Big[\frac{m\,V\,[\,(1\,-\,\sqrt{3}\,\alpha)\,\widehat{x} + \widehat{y} + \widehat{z}]}{\sqrt{2\,+\,(1\,-\,\sqrt{3}\,\alpha)^2}}\Big] \;-\; \Big[\frac{m\,V\,(\,\widehat{y}\,+\,\widehat{z}\,)}{\sqrt{(1-\alpha)^2\,+\,2}}\Big]$$

$$\equiv\; A_0\,\widehat{x} \;+\; B_0\,(\,\widehat{y}\,+\,\widehat{z}\,)$$

(b.) The angular momentum at the instant before the collision

$$\vec{L}_T = \;\vec{r}_C\;\times\;\vec{P}_T\;=\;\frac{-\sqrt{3}\,\alpha\,m\,V\,R\,[\,\widehat{x} + \widehat{y} + \widehat{z}]\;\times\,\widehat{x}}{\sqrt{3[\,2\,+\,(1\,-\,\sqrt{3}\,\alpha)^2\,]}}$$

$$=\;\Big[\frac{-\sqrt{3}\,\alpha\,m\,V\,R}{\sqrt{3[\,2\,+\,(1\,-\,\sqrt{3}\,\alpha)^2\,]}}\Big]\,(\,\widehat{y}\,-\,\widehat{z}\,)\;=\;C_0\,(\,\widehat{y}\,-\,\widehat{z}\,)$$

The angular momentum of the ball after the collision is

$$\vec{L}_b = \;\vec{r}_C\;\times\;\vec{P}_b\;=\;m\,V\,R\,\Big[\frac{\widehat{x}\;\times\;(\,\widehat{y}\,+\,\widehat{z}\,)}{\sqrt{3\,[\,(1-\alpha)^2\,+\,2\,]}}\Big]$$

$$=\;\Big[\frac{m\,V\,R}{\sqrt{3\,[\,(1-\alpha)^2\,+\,2\,]}}\Big]\,(\,\widehat{y}\,-\,\widehat{z}\,)\;=\;D_0\,(\,\widehat{y}\,-\,\widehat{z}\,)\quad .$$

Conservation of angular momentum implies the angular momentum of the sphere $\vec{L}_G$ is given by

$$\vec{L}_G\;=\;\vec{L}_T\;-\;\vec{L}_b\;=\;\sqrt{2}\,[\,C_0\;-\;D_0\,]\,\frac{1}{\sqrt{2}}\,(\,\widehat{y}\,-\,\widehat{z}\,)$$

and using the moment of inertia of the globe

$$\vec{L}_G\;=\;\frac{2}{3}\,M\,R^2\,\vec{\omega}\;=\;\frac{2}{3}\,M\,R^2\,\omega\,\widehat{\omega}\quad .$$

These different expressions are set equal one to the other to find $\widehat{\omega} = \frac{1}{\sqrt{2}}\,(\,\widehat{y}\,-\,\widehat{z}\,)$ and

$$\omega\;=\;\frac{3}{\sqrt{2}\,M\,R^2}\,[\,C_0\;-\;D_0\,]$$

Problem (4.)

The earth is actually slowing down in its spinning about its axis. Let us model this by writing

$$\vec{\Omega} \;=\; \Omega_0\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;][\;sin\lambda\,\hat{r} \;-\; cos\lambda\,\hat{\theta}\;] \qquad ,$$

where $\Omega_0 = 7.5 \times 10^{-5} s^{-1}$ and $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are the standard unit vectors of a spherical coordinate system.

(a.) As was noted in class one day, the actual form of Newton's Second law for a rotating frame is given by

$$m\,\ddot{\vec{r}} \;=\; \vec{F} \;+\; 2\,m\,\dot{\vec{r}} \times \vec{\Omega} \;+\; m\,\vec{r} \times \dot{\vec{\Omega}} \;+\; m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega}$$

The second term is velocity-dependent and is the "Coriolis Term" which is to be ignored in our analysis following the text and $\vec{F}$ corresponds to force of gravity on the surface of the Earth $\vec{F} = -\,m\,g_0\,\hat{r}$. Since $\mathcal{H} << R_E$, $\vec{r} \sim R_E\,\hat{r}$ which implies

$$\begin{aligned}
m\,\ddot{\vec{r}} = \;& -\,m\,g_0\,\hat{r} \;-\; m\,\Omega_0\,(\frac{\alpha_0}{\O_0})\,R_E\,\hat{r} \,\times\, [\;sin\lambda\,\hat{r} \;-\; cos\lambda\,\hat{\theta}\;] \\
& +\; m\,(\Omega_0)^2\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,\Big(\,[\,sin\lambda\,\hat{r} \;-\; cos\lambda\,\hat{\theta}] \,\times\, \hat{r}\,\Big) \times \\
& [\,sin\lambda\,\hat{r} \;-\; cos\lambda\,\hat{\theta}] \\
= \;& -\,m\,g_0\,\hat{r} \;+\; m\,\Omega_0\,(\frac{\alpha_0}{\O_0})\,R_E\,cos\lambda\,\hat{\phi} \\
& +\; m\,(\Omega_0)^2\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,R_E\,cos\lambda\,[\,cos\lambda\,\hat{r} \;+\; sin\lambda\,\hat{\theta}] \\
= \;& -\,m\,\Big[\,g_0 \;-\; (\Omega_0)^2\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,R_E\,cos^2\lambda\,\Big]\,\hat{r} \\
& +\; m\,(\Omega_0)^2\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,R_E\,cos\lambda\,sin\lambda\,\hat{\theta} \\
& +\; m\,\Omega_0\,(\frac{\alpha_0}{\O_0})\,R_E\,cos\lambda\,\hat{\phi} \;=\; m\,\vec{g}_{eff} \qquad .
\end{aligned}$$

This implies upon using the local x-y-z coordinates

$$\begin{aligned}
\vec{g}_{eff} = \;& \Omega_0\,(\frac{\alpha_0}{\O_0})\,R_E\,cos\lambda\,\hat{x} \;-\; (\Omega_0)^2\,R_E\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,cos\lambda\,sin\lambda\,\hat{y} \\
& -\;\Big[\,g_0 \;-\; (\Omega_0)^2\,R_E\,[\,1 \;-\; (\frac{\alpha_0}{\O_0})t\;]^2\,cos^2\lambda\,\Big]\,\hat{z} \qquad .
\end{aligned}$$

$$\frac{d^2 x}{dt^2} = \Omega_0 \, (\frac{\alpha_0}{\o_0}) \, R_E \, cos\lambda$$

$$\frac{d^2 y}{dt^2} = - \; (\Omega_0)^2 \, R_E \, cos\lambda \, sin\lambda \, [\, 1 \; - \; (\frac{\alpha_0}{\o_0}) t \,]^2$$

$$\frac{d^2 z}{dt^2} = - \Big[\, g_0 \; - \; (\Omega_0)^2 \, R_E \, cos^2\lambda \, [\, 1 \; - \; (\frac{\alpha_0}{\o_0}) t \,]^2 \Big]$$

This gives the following equations for the position of the object.

$$x(t) = \; \tfrac{1}{2} \, \Omega_0 \, (\frac{\alpha_0}{\o_0}) \, R_E \, cos\lambda \, t^2$$

$$y(t) = \; - \; \frac{R_E \, cos\lambda \, sin\lambda}{12} \, (\frac{\Omega_0 \, \o_0}{\alpha_0})^2 \Big\{ \, [\, 1 \; - \; (\frac{\alpha_0}{\o_0}) t \,]^4 \; + \; 4 \, (\frac{\alpha_0}{\o_0}) \, t \; - \; 1 \, \Big\}$$

$$z(t) = \; \mathcal{H} \; - \; \tfrac{1}{2} \, g_0 \, t^2 \; + \; \frac{R_E \, cos^2\lambda}{12} \, (\frac{\o_0 \, \Omega_0}{\alpha_0})^2 \Big\{ \, [\, 1 \; - \; (\frac{\alpha_0}{\o_0}) t \,]^4 \; + \; 4 \, (\frac{\alpha_0}{\o_0}) \, t \; - \; 1 \, \Big\}$$

Thus an object that starts at position $(0, 0, \mathcal{H})$ lands at

$$\vec{r}_f \;\approx\; \Omega_0 \, R_E \, [\, (\frac{\alpha_0}{\o_0}) \, cos\lambda \, \hat{x} \; - \; (\Omega_0) \, cos\lambda \, sin\lambda \, \hat{y} \,] \Big[\frac{\mathcal{H}}{g_0} \Big] \quad .$$

where we have approximated the time it takes to land by $\sqrt{2 \, \mathcal{H}/g_0}$ and neglected terms of order $(\o_0/\alpha_0)^2$.

Problem (5.)

(a.) Along the equator $z = 0$ the force $\vec{F}$ given by

$$\vec{F} \;=\; \Big[\frac{f_2 \, [\, x^3 \, \widehat{x} \; - \; x \, z^2 \, \widehat{y} \; - \; 2 \, x \, y \, z \, \widehat{z} \,]}{(R)^3} \Big] \quad .$$

is much more simply expressed by

$$\vec{F} \;=\; \frac{f_2}{R^3} \, x^3 \, \widehat{x}$$

and the path is described by (with $0 \le \phi \le \pi$)

$$\vec{\ell}(\phi) \;=\; R \Big[cos\phi \, \widehat{x} \; + \; sin\phi \, \widehat{y} \Big] \quad \longrightarrow \quad x \;=\; R \, cos\phi \quad , \quad y \;=\; R \, sin\phi$$

$$d\vec{\ell} \;=\; R \, d\phi \Big[\; - \; sin\phi \, \widehat{x} \; + \; cos\phi \, \widehat{y} \Big]$$

This all implies that

$$\begin{aligned}\int_0^{\pi} \vec{F}\cdot d\vec{\ell} = & \; - \; f_2\, R \int_0^{\pi} d\phi \, cos^3\phi \, sin\phi \, d\phi \\ = & \; - \; f_2\, R \int_{-1}^{1} du \, u^3 \; du \;=\; 0.\end{aligned}$$

(b.) The shortest path on a sphere is a circle. We need a circle that includes the points described by the vectors

$$\vec{\xi}_1 \;=\; R\,\widehat{x} \;\;,\;\; \vec{\xi}_2 \;=\; \frac{1}{\sqrt{3}}R[\,\widehat{x}+\widehat{y}+\widehat{z}] \;\;,\;\; \vec{\xi}_3 \;=\; -\;R\,\widehat{x} \;\;.$$

We start by finding the normal to the plane that contains these three vectors. This is found by calculating the cross product between $\vec{\xi}_1$ and $\vec{\xi}_2$ then writing the unit vector $\widehat{N}$ that points along the direction of the cross product. This vector is given by

$$\widehat{N} \;=\; \frac{1}{\sqrt{2}}[\,-\,\widehat{y} \;+\; \widehat{z}\,]$$

If we cross this vector with $\widehat{x}$, we must get a vector

$$\widehat{M} \;=\; \frac{1}{\sqrt{2}}[\,\widehat{y} \;+\; \widehat{z}\,]$$

that must also lie in the plane along with $\vec{\xi}_1$ and $\vec{\xi}_2$. It must be the case that the path for which we are looking takes the form,

$$\begin{aligned}\vec{\ell}(\psi) \;=&\; R\Big[\,cos\psi\,\widehat{x} \;+\; sin\psi\,\widehat{M}\,\Big] \;\;\rightarrow \\ & R\Big[\,cos\psi\,\widehat{x} \;+\; \frac{1}{\sqrt{2}}sin\psi\,\widehat{y} \;+\; \frac{1}{\sqrt{2}}sin\psi\,\widehat{z}\,\Big] \;\;\rightarrow \\ d\vec{\ell} \;=&\; R\,d\psi\Big[\,-\;sin\psi\,\widehat{x} \;+\; \frac{1}{\sqrt{2}}cos\psi\,\widehat{y} \;+\; \frac{1}{\sqrt{2}}cos\psi\,\widehat{z}\,\Big] \\ x \;=&\; R\,cos\psi \;\;\;,\;\;\; y \;=\; \frac{1}{\sqrt{2}}R\,sin\psi \;\;,\;\; z \;=\; \frac{1}{\sqrt{2}}R\,sin\psi\end{aligned}$$

Notice that for the path described by $\vec{\ell}(\psi)$

$$cos\psi \;=\; \frac{1}{\sqrt{3}} \;\;\rightarrow\;\; sin\psi \;=\; \sqrt{\,1\;-\;cos^2\psi} \;\;\rightarrow\;\; sin\psi \;=\; \sqrt{\frac{2}{3}}$$

so that when $\psi \;=\; tan^{-1}\sqrt{2}$ we find

$$\vec{\ell}(tan^{-1}\sqrt{2}) = \frac{1}{\sqrt{3}}\,R\Big[\,\widehat{x} \;+\; \widehat{y} \;+\; \widehat{z}\,\Big]$$

and this is the point thru this path must pass. Along this path the force, $\vec{F}$ is given by

$$\vec{F} \;=\; f_2\,[\;cos^3\psi\,\widehat{x} \;-\; \frac{1}{2}\,cos\psi\,sin^2\psi\,\widehat{y} \;-\; cos\psi\,sin^2\psi\,\widehat{z}\;] \quad .$$

This all implies that

$$
\begin{aligned}
\int_0^{\pi} \vec{F} \cdot d\vec{\ell} &= \; - \; f_2 \, R \int_0^{\pi} d\psi \left[cos^3\psi \, sin\psi \; + \; \frac{3}{2\sqrt{2}} \, cos^2\psi \, sin^2\psi \right] \\
&= \; - \; \frac{3}{2\sqrt{2}} \, f_2 \, R \int_0^{\pi} d\psi \left[cos^2\psi \, sin^2\psi \right] \\
&= \; - \; \frac{3}{8\sqrt{2}} \, f_2 \, R \int_0^{\pi} d\psi \left[\, sin^2(2\,\psi) \right] \\
&= \; - \; \frac{3}{16\sqrt{2}} \, f_2 \, R \int_0^{\pi} d\psi \left[1 \; - \; cos(4\,\psi) \right] \; = \; - \; \frac{3\,\pi}{16\sqrt{2}} \, f_2 \, R
\end{aligned}
$$