HW 6 soln.

10-9. We adopt the coordinate system as in eg. 10.3 in book, so

$$\omega_x = -\omega\cos\lambda$$
$$\omega_y = 0$$
$$\omega_z = \omega\sin\lambda.$$

$$\dot{x} \simeq 0$$
$$\dot{y} = V_0\cos\alpha$$
$$\dot{z} = V_0\sin\alpha - gt.$$

and

$$\vec{\omega}\times\vec{V}_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & V_0\cos\alpha & V_0\sin\alpha - gt \end{vmatrix}$$

$$= V_0\cos\alpha\,\omega\sin\lambda\,\hat{x} + \omega\cos\lambda(V_0\sin\alpha - gt)\hat{y} + V_0\cos\alpha\,\omega\cos\lambda\,\hat{z}$$

hence $\vec{a}_r = -g\hat{z} + 2\vec{\omega}\times\vec{V}_r$

$$a_{rx} = 2V_0\omega\cos\alpha\sin\lambda.$$

time for projectile to travel is given by

$$0 = V_0\sin\alpha\, t - \frac{1}{2}gt^2$$

$$t = \frac{2V_0\sin\alpha}{g}.$$

hence deflection in x-dir

$$d = \frac{1}{2} a_{rx} t^2$$

$$= \frac{1}{2}(2V_0\omega \cos\alpha \sin\lambda)\left(\frac{2V_0 \sin\alpha}{g}\right)^2$$

$$= \frac{4V_0^3}{g} \omega \sin\lambda \sin^2\alpha \cos\alpha$$

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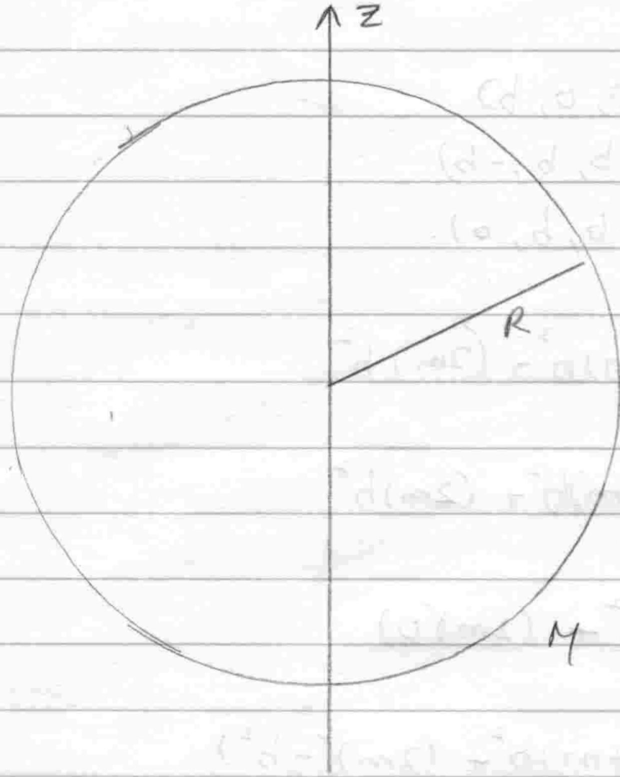
11.1.



density of sphere $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$

$$I_z = \int dm\,(r^2 - z^2) = \rho \int dV\,(r^2 - z^2)$$

$$= \rho \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^R r^2 dr\,(r^2 - r^2\cos^2\theta)$$

$$= 2\pi\rho \left(\int_0^{\pi} d\theta\,\sin^3\theta\right)\left(\int_0^R r^4 dr\right)$$

$$= 2\pi\rho \left[\int_{-1}^{1} d\alpha\,(1-\alpha^2)\right] \frac{R^5}{5} \qquad \text{where } \alpha = \cos\theta$$

$$= 2\pi\rho \frac{R^5}{5} \cdot \frac{4}{3} = \frac{2}{5} M R^2.$$

Since the system is spherically symmetric obviously we have

$$I_x = I_y = I_z = \frac{2}{5} M R^2$$

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1H3. $m_1 = 3m$, $(b, 0, b)$

$m_2 = 4m$, $(b, b, -b)$

$m_3 = 2m$, $(-b, b, 0)$.

$$\sum_i m_i x_i^2 = (3m)b^2 + (4m)b^2 + (2m)b^2 = 9mb^2$$

$$\sum_i m_i y_i^2 = (3m)(0) + (4m)b^2 + (2m)b^2 = 6mb^2.$$

$$\sum_i m_i z_i^2 = (3m)b^2 + 4mb^2 + (2m)(0) = 7mb^2.$$

$$\sum_i m_i x_i y_i = (3m)(0) + (4m)b^2 + (2m)(-b^2) = 2mb^2$$

$$\sum_i m_i x_i z_i = (3m)b^2 + (4m)(-b^2) + (2m)(0) = -mb^2.$$

$$\sum_i m_i y_i z_i = (3m)(0) + (4m)(-b^2) + (2m)(0) = -4mb^2.$$

and $\sum_i m_i (x_i^2 + y_i^2 + z_i^2) = (9+6+7)mb^2 = 22mb^2$.

$$\underline{I} = 22mb^2 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 9mb^2 & 2mb^2 & -mb^2 \\ 2mb^2 & 6mb^2 & -4mb^2 \\ -mb^2 & -4mb^2 & 7mb^2 \end{bmatrix}$$

$$= mb^2 \begin{bmatrix} 13 & -2 & +1 \\ -2 & 16 & +4 \\ +1 & +4 & 15 \end{bmatrix}$$

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diagonalize $\underline{I}$

$$|\lambda \mathbb{1} - \underline{I}| = 0$$

$$\begin{vmatrix} \lambda - 13mb^2 & 2mb^2 & -mb^2 \\ 2mb^2 & \lambda - 16mb^2 & -4mb^2 \\ -mb^2 & -4mb^2 & \lambda - 15mb^2 \end{vmatrix} = 0 .$$

$$(\lambda - 13mb^2)\left[(\lambda - 16mb^2)(\lambda - 15mb^2) - 16(mb^2)^2\right]$$
$$- 2mb^2\left[2mb^2(\lambda - 15mb^2) - 4(mb^2)^2\right]$$
$$- mb^2\left[2mb^2(-4mb^2) + mb^2(\lambda - 16mb^2)\right] = 0 .$$

$$(\lambda - 13mb^2)\left[\lambda^2 - 31mb^2\lambda + 224(mb^2)^2\right]$$
$$- 2mb^2\left[2mb^2\lambda - 34(mb^2)^2\right]$$
$$- mb^2\left[mb^2\lambda - 24(mb^2)^2\right] = 0$$

$$\lambda^3 - 31mb^2\lambda^2 + 224(mb^2)^2\lambda - 13mb^2\lambda^2 + 403(mb^2)\lambda - 2912(mb^2)^3$$
$$- 4(mb^2)^2\lambda + 68(mb^2)^3$$
$$- (mb^2)^2\lambda + 24(mb^2)^3 = 0$$

$$\lambda^3 - 44(mb^2)\lambda^2 + 622(mb^2)^2\lambda - 2820(mb^2)^3 = 0 .$$

the solns are

$$I_1 = 10mb^2 .$$
$$I_2 = (17 - \sqrt{7})mb^2$$
$$I_3 = (17 + \sqrt{7})mb^2 .$$

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where I have denoted the solns. for λ by the I's.

Sub. the I_1, I_2, I_3 Individually in the system of eqns

$$\left[\begin{array}{ccc|c} I_i - 13mb^2 & 2mb^2 & -mb^2 & 0 \\ 2mb^2 & I_i - 16mb^2 & -4mb^2 & 0 \\ -mb^2 & -4mb^2 & I_i - 15mb^2 & 0 \end{array}\right]$$ where $i = 1, 2, 3$.

and solving, get the eigenvectors

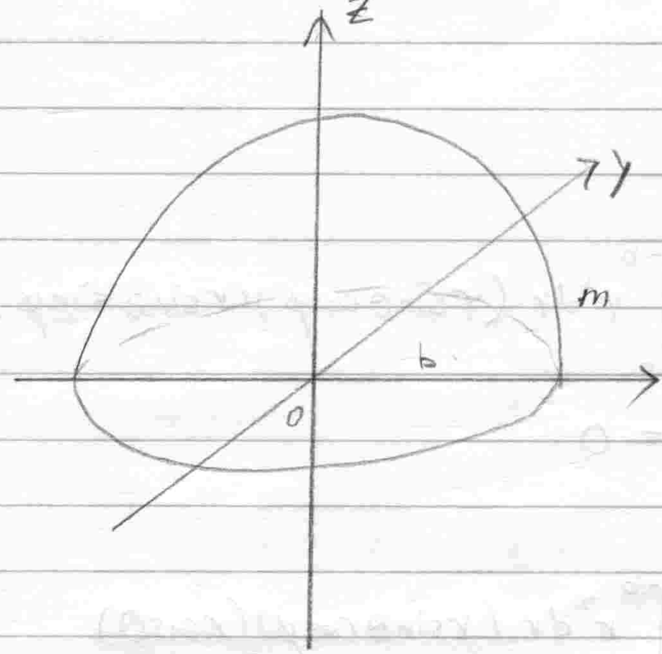
$$|I_1\rangle = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$|I_2\rangle = \begin{bmatrix} \frac{1}{3}(2+\sqrt{7}) \\ \frac{1}{3}(1-\sqrt{7}) \\ 1 \end{bmatrix}$$

$$|I_3\rangle = \begin{bmatrix} \frac{1}{3}(2-\sqrt{7}) \\ \frac{1}{3}(1+\sqrt{7}) \\ 1 \end{bmatrix}$$

they are the principal axes.

11-14.



it would be convenient to compute the principal axes & principal moments of inertia about the point o first, then translate the axes to get the desired results by // axis thm.

density $\rho = \dfrac{m}{\frac{2}{3}\pi b^3}$

$$\int dV x^2 = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \int_0^b r^2 dr \cdot (r\sin\theta\cos\phi)^2$$

$$= \left(\int_0^{2\pi} d\phi \frac{1+\cos 2\phi}{2}\right)\left[\int_0^1 d\alpha(1-\alpha^2)\right]\left[\frac{b^5}{5}\right] \quad \text{where } \alpha = \cos\theta$$

$$= \pi \cdot \frac{2}{3} \frac{b^5}{5} = \frac{2\pi}{15} b^5$$

$$\int dV y^2 = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \int_0^b r^2 dr\,(r\sin\theta\sin\phi)^2$$

$$= \frac{2\pi}{15} b^5$$

$$\int dV z^2 = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \int_0^b r^2 dr (r\cos\theta)^2 = 2\pi\left(\int_0^1 d\alpha\, \alpha^2\right)\frac{b^5}{5}$$

$$= \frac{2\pi}{15} b^5$$

$$\int dv\, xy = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^b r^2 dr\, (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)$$

$$\propto \int_0^{2\pi} \sin 2\phi \, d\phi = 0.$$

$$\int dv\, xz = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^b r^2 dr\, (r\sin\theta\cos\phi)(r\cos\theta)$$

$$\propto \int_0^{2\pi} \cos\phi \, d\phi = 0$$

$$\int dv\, yz = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^b r^2 dr\, (r\sin\theta\sin\phi)(r\cos\theta)$$

$$\propto \int_0^{2\pi} \sin\phi \, d\phi = 0.$$

So by choosing the x, y, z axes as shown we ~~are~~ have obtained automatically the correct principal axes about point O.

$$\underline{I} = \begin{bmatrix} \frac{4\pi}{15} b^5 & & \\ & \frac{4\pi}{15} b^5 & \\ & & \frac{4\pi}{15} b^5 \end{bmatrix} \frac{m}{\frac{4}{3}\pi b^3} = \frac{2}{5} b^2 m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the CM is,

$$\vec{\zeta} = \frac{\rho \int dV \, \vec{r}}{\rho \int dV}$$

By symmetry the CM should lie on z-axis, so

$$\zeta = \frac{1}{V} \int dV \, z .$$

and

$$\int dV \, z = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \int_0^b r^2 dr \cdot (r\cos\theta)$$

$$= 2\pi \left(\int_0^1 d\alpha \, \alpha \right) \left(\frac{b^4}{4} \right)$$

$$= \frac{\pi b^4}{4} .$$

$$\zeta = \frac{1}{\frac{2}{3}\pi b^3} \frac{\pi b^4}{4} = \frac{3b}{8} .$$

apply // axis thm,

$$I_{CM,z} = \frac{2}{5} m b^2$$

#

$$I_{CM,x} = \frac{2}{5} m b^2 - m\left(\frac{3b}{8}\right)^2 = \frac{83}{320} m b^2$$

#

$$I_{CM,y} = \frac{2}{5} m b^2 - m\left(\frac{3b}{8}\right)^2 = \frac{83}{320} m b^2$$

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HW 7 Jala

1. The Euler's eqns are

$$I_1 \dot\omega_1 = (I_2 - I_3)\,\omega_2\omega_3$$
$$I_2 \dot\omega_2 = (I_3 - I_1)\,\omega_3\omega_1$$
$$I_3 \dot\omega_3 = (I_1 - I_2)\,\omega_1\omega_2 .$$

with $\dfrac{I_1}{I_3} = \dfrac{1}{4}$, $\dfrac{I_2}{I_3} = \dfrac{1}{2}$

$$\begin{cases} \dot\omega_1 = -2\,\omega_2\omega_3 . \\ \dot\omega_2 = \dfrac{3}{2}\,\omega_3\omega_1 \\ \dot\omega_3 = -\dfrac{1}{4}\,\omega_1\omega_2 . \end{cases} \quad , \text{ I.C. } \frac{\omega_1}{\omega_3} = 0.2 ; \ \frac{\omega_2}{\omega_3} = 0 .$$

these are to be solved with Mathematica by taking $\omega_3 = 1$.

For slight deviations from the equil. pt. $\omega_3 \neq 0$, $\omega_1 = \omega_2 = 0$ and $\omega_3 \gg \omega_1, \omega_2$.

$\omega_3 = \text{const}$.

$$\ddot\omega_1 = -\frac{\omega_3^2 (I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\,\omega_1 . = -\Omega^2 \omega_1$$

$$\ddot\omega_2 = -\frac{\omega_3^2 (I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\,\omega_2 = -\Omega^2 \omega_2 .$$

$$\Omega^2 = \omega_3^2 \left(\frac{I_3}{I_2} - 1\right)\left(\frac{I_3}{I_1} - 1\right) = \omega_3^2 (2-1)(4-1) = 3\omega_3^2 .$$

$\Omega = \sqrt{3}$ taking $\omega_3 = 1$.

$$\dot{w}_2(t) = A\sqrt{3}\cos(\sqrt{3}t)$$

$$\dot{w}_2(0) = A\sqrt{3} = 0.3 \Rightarrow A = \frac{0.3}{\sqrt{3}}$$

With $w_1(0) = 0.2$, $\dot{w}_1(0) = -2(0) = 0$

$$w_1(t) = B\cos(\sqrt{3}t).$$

$$w_1(0) = B = 0.2.$$

$\therefore$ Solns are

$$\begin{cases} w_1(t) = 0.2\cos(\sqrt{3}t) \\ w_2(t) = \dfrac{0.3}{\sqrt{3}}\sin(\sqrt{3}t). \\ w_3(t) = 1 \end{cases}$$

```
Omega =
 NDSolve[{w1'[t] == -2 w2[t] w3[t], w2'[t] == 1.5 w3[t] w1[t], w3'[t] == -0.25 w1[t] w2[t],
   w1[0] == 0.2, w2[0] == 0, w3[0] == 1}, {w1, w2, w3}, {t, 0, 10}]

{{w1 → InterpolatingFunction[{{0., 10.}}, <>],
  w2 → InterpolatingFunction[{{0., 10.}}, <>],
  w3 → InterpolatingFunction[{{0., 10.}}, <>]}}

"Analytical solutions"
```

wa1[t] = 0.2 Cos$[\sqrt{3}\ t]$

0.2 Cos$[\sqrt{3}\ t]$

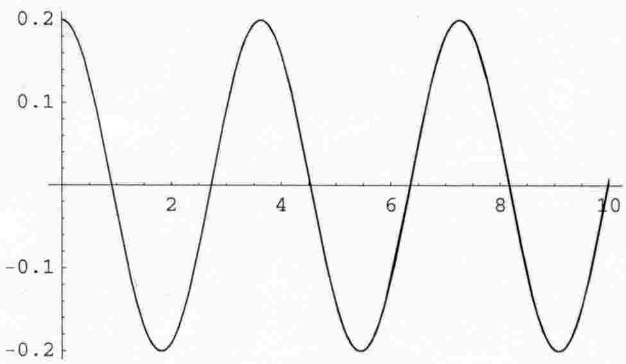
wa2[t] = $\frac{0.3}{\sqrt{3}}$ Sin$[\sqrt{3}\ t]$

0.173205 Sin$[\sqrt{3}\ t]$

```
wa3[t] = 1

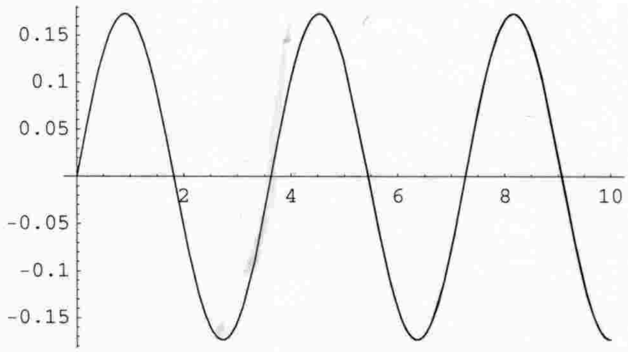
1

Plot[Evaluate[{w1[t] /. Omega, wa1[t]}], {t, 0, 10}]
```



- Graphics -

```
Plot[Evaluate[{w2[t] /. Omega, wa2[t]}], {t, 0, 10}]
```



- Graphics -

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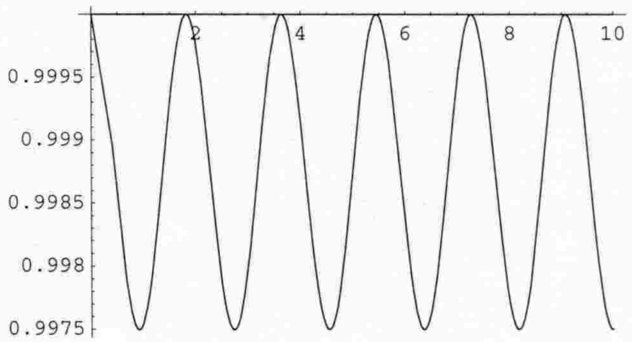
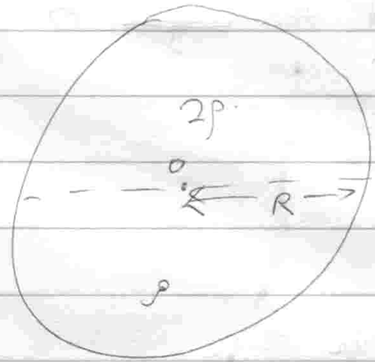
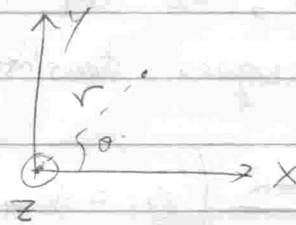
Plot[Evaluate[{w3[t] /. Omega, wa3[t]}], {t, 0, 10}]
```



- Graphics -

11-25





$$\text{CM} \quad \vec{\zeta} = \frac{\int dm\, \vec{r}}{\int dm}$$

by symmetry the CM lies on the y-axis, so

$$= \frac{\int dm\, y}{\int dm}$$

it would be convenient to employ polar coordinates in this problem,

$$\text{so} \quad \zeta = \frac{\int dm\, r \sin\theta}{\int dm}$$

$$= \frac{1}{\frac{\pi R^2}{2}(2\rho) + \frac{\pi R^2}{2}(\rho)} \left\{ 2\rho \int_0^{\pi} d\theta \int_0^R r\,dr + \rho \int_{\pi}^{2\pi} d\theta \int_0^R r\,dr \right\} r\sin$$

$$= \frac{1}{\left(\frac{3\rho\pi R^2}{2}\right)} \left\{ 4\rho \frac{R^3}{3} + \rho(-2)\frac{R^3}{3} \right\} = \frac{4R}{9\pi}.$$

as the disc rotates, the CM rotates about the ~~origin~~ centre O while the ~~origin~~ centre itself undergoes translational motion,

$$CM \quad \vec{\xi}(t) = \vec{r}_0 + \frac{4R}{9\pi}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

where $\vec{r}_0$ is the position of the centre of the disc. and it's assumed the disc rotates in the $-\hat{x}$ direction.

thus $$\frac{d\vec{\xi}(t)}{dt} = \frac{d\vec{r}_0}{dt} + \frac{4R}{9\pi}\dot{\theta}\left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)$$

for rolling w/o slipping, centre moves with velocity $\frac{d\vec{r}_0}{dt} = -R\dot{\theta}\hat{x}$, thus

$$\vec{V}_{CM} \equiv \frac{d\vec{\xi}(t)}{dt} = -R\dot{\theta}\hat{x} + \frac{4R}{9\pi}\dot{\theta}\left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)$$

translational KE is

$$T_{CM} = \frac{1}{2}\left(\frac{3\rho\pi R^2}{2}\right)\left|\vec{V}_{CM}\right|^2$$

$$= \frac{3\rho\pi R^2}{4}\left\{\left(R\dot{\theta} + \frac{4R}{9\pi}\dot{\theta}\sin\theta\right)^2 + \left(\frac{4R}{9\pi}\dot{\theta}\cos\theta\right)^2\right\}$$

$$= \frac{3\rho\pi R^2}{4}\left\{R^2\dot{\theta}^2 + \frac{8R^2\dot{\theta}^2}{9\pi}\sin\theta + \left(\frac{4R}{9\pi}\dot{\theta}\right)^2\right\}$$

$$= \frac{3\rho\pi R^2}{4}\left\{\left(1 + \frac{16}{81\pi^2}\right)R^2\dot{\theta}^2 + \frac{8R^2\dot{\theta}^2}{9\pi}\sin\theta\right\}$$

$$= \frac{3\rho\pi R^4}{4}\dot{\theta}^2\left\{\left(1 + \frac{16}{81\pi^2}\right) + \frac{8}{9\pi}\sin\theta\right\}$$

$$I_z = \int dm\, r^2$$

$$= \left\{2\rho\int_0^{\pi} d\theta \int_0^{R} r\,dr + \rho\int_{\pi}^{2\pi} d\theta \int_0^{R} r\,dr\right\} r^2$$

$$= 2\rho\pi\frac{R^4}{4} + \rho\pi\frac{R^4}{4} = \frac{3\rho\pi R^4}{4} \qquad \left(=\frac{1}{2}MR^2\right)$$

by // axis thm, moment of inertia about CM is

$$I_{CM} = I_z - \left(\frac{4R}{9\pi}\right)^2\left(\frac{3\rho\pi R^2}{2}\right)$$

$$= \frac{3\rho\pi R^4}{4}\left(1 - \frac{32}{81\pi^2}\right)$$

thus rotational KE

$$T_{rot} = \frac{1}{2} I_{CM}\,\dot\theta^2 = \frac{3\rho\pi R^4}{8}\dot\theta^2\left(1 - \frac{32}{81\pi^2}\right)$$

finally, the PE is

$$V = \left(\frac{3\rho\pi R^2}{2}\right)\left(\frac{4R}{9\pi}\right) g\sin\theta$$

$$= \frac{2\rho R^3}{3} g\sin\theta$$

Lagrangian

$$L = T_{tran} + T_{rot} - V$$

$$= \frac{3\rho\pi R^4}{4}\dot{\theta}^2\left[\left(1+\frac{16}{81\pi^2}\right)+\frac{8}{9\pi}\sin\theta\right]$$

$$+ \frac{3\rho\pi R^4}{8}\dot{\theta}^2\left(1-\frac{32}{81\pi^2}\right) - \frac{2\rho R^3}{3} g\sin\theta .$$

~~or in terms of total mass $M = \frac{3\rho\pi R^2}{2}$,~~

$$L = \frac{3\rho\pi R^4}{8}\dot{\theta}^2\left(3+\frac{16}{9\pi}\sin\theta\right) - \frac{2\rho R^3}{3} g\sin\theta .$$

$\#$

or in terms of total mass $M = \frac{3\rho\pi R^2}{2}$,

$$= \frac{1}{2}MR^2\dot{\theta}^2\left(3+\frac{16}{9\pi}\sin\theta\right) - \frac{4MgR}{9\pi}\sin\theta .$$

$\#$

11-28. from Fig 11-9

We can write $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ in component forms in the x_i'-coordinates,

$$[\dot{\phi}]' = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$[\dot{\theta}]' = \begin{bmatrix} \dot{\theta}\cos\phi \\ \dot{\theta}\sin\phi \\ 0 \end{bmatrix}$$

$$[\dot{\psi}]' = \begin{bmatrix} \dot{\psi}\sin\theta\sin\phi \\ -\dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\cos\theta \end{bmatrix}$$

hence $$[\omega]' = [\dot{\phi}]' + [\dot{\theta}]' + [\dot{\psi}]'$$

$$= \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}$$ 本.

11-31. Euler's eqns.

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 .$$

$$I_1 = I_2\cos 2\alpha = I\cos 2\alpha \qquad \text{where } I_2 \equiv I .$$
$$I_3 = I\cos 2\alpha + I = I(1+\cos 2\alpha)$$

Sub. into Euler's eqns

$$I\cos 2\alpha\,\dot{\omega}_1 = -I\cos 2\alpha\,\omega_2\omega_3 .$$
$$I\dot{\omega}_2 = I\,\omega_3\omega_1$$
$$I(1+\cos 2\alpha)\dot{\omega}_3 = I(\cos 2\alpha - 1)\omega_1\omega_2 .$$

or
$$\dot{\omega}_1 = -\omega_2\omega_3 . \qquad \text{—— (1)}$$
$$\dot{\omega}_2 = \omega_3\omega_1 . \qquad \text{—— (2)}$$
$$\dot{\omega}_3 = \frac{\cos 2\alpha - 1}{\cos 2\alpha + 1}\omega_1\omega_2 .$$
$$= -\tan^2\alpha\,\omega_1\omega_2 . \qquad \text{—— (3)}$$

$(1)\times\omega_1 + (2)\times\omega_2$

$$\Rightarrow \omega_1\dot{\omega}_1 = -\omega_2\dot{\omega}_2$$

$$\frac{d\omega_1^2}{dt} = -\frac{d\omega_2^2}{dt} .$$

With the I.C., $\omega_1(0) = \Omega\cos\alpha$, $\omega_2(0) = 0$, $\omega_3(0) = \Omega\sin\alpha$

$$\int_0^t \frac{d\omega_1^2}{dt}\,dt = -\int_0^t \frac{d\omega_2^2}{dt}\,dt$$

$$\omega_1^2(t) - \omega_1^2(0) = -\omega_2^2(t) + \omega_2^2(0)$$

$$\omega_1(t) = \sqrt{\omega_1^2(0) - \omega_2^2(t)}\,.$$

$$\dot\omega_3\omega_3 = -\tan^2\alpha\,\omega_1\omega_2\omega_3$$

$$= -\tan^2\alpha\,\omega_2\dot\omega_2 \qquad \text{by (2)}$$

$$\frac{d\omega_3^2}{dt} = -\tan^2\alpha\,\frac{d\omega_2^2}{dt}$$

$$\omega_3^2(t) - \omega_3^2(0) = -\tan^2\alpha\left[\omega_2^2(t) - \omega_2^2(0)\right].$$

$$\omega_3(t) = \sqrt{\omega_3^2(0) - \tan^2\alpha\,\omega_2^2(t)}$$

hence $\dot\omega_2 = \omega_3\omega_1$ — (2)

$$\dot\omega_2^2 = \left[\omega_1^2(0) - \omega_2^2(t)\right]\left[\omega_3^2(0) - \tan^2\alpha\,\omega_2^2(t)\right].$$

$$= \left[\Omega^2\cos^2\alpha - \omega_2^2(t)\right]\left[\Omega^2\sin^2\alpha - \tan^2\alpha\,\omega_2^2(t)\right]$$

$$= \tan^2\alpha\left[\Omega^2\cos^2\alpha - \omega_2^2(t)\right]^2.$$

$$\Rightarrow \dot\omega_2 = \tan\alpha\left[\Omega^2\cos^2\alpha - \omega_2^2(t)\right].$$

Sub. $\omega_2(t) = \Omega\cos\alpha\tanh(\Omega t\sin\alpha)$ as given in book,

$$LHS = \dot\omega_2 = \Omega\cos\alpha\,\mathrm{sech}^2(\Omega t\sin\alpha)\,\Omega\sin\alpha$$

$$= \Omega^2\sin\alpha\cos\alpha\,\mathrm{sech}^2(\Omega t\sin\alpha)$$

$$RHS = \tan\alpha\left[\Omega^2\cos^2\alpha - \Omega^2\cos^2\alpha\tanh^2(\Omega t\sin\alpha)\right]$$

$$= \Omega^2\sin\alpha\cos\alpha\left[1 - \tanh^2(\Omega t\sin\alpha)\right] = \Omega^2\sin\alpha\cos\alpha\,\mathrm{sech}^2(\Omega t\sin\alpha)$$

$$= LHS$$

#.

HW 8 Soln

3-8. $\begin{cases} x = a(\phi - \sin\phi) \\ y = a(\cos\phi - 1) \end{cases}$

$$\frac{dx}{d\phi} = a(1-\cos\phi) \quad , \quad \frac{dy}{d\phi} = -a(\sin\phi).$$

$$\frac{ds}{d\phi} = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2}.$$

$$= \sqrt{a^2(1 - 2\cos\phi + \cos^2\phi + \sin^2\phi)}$$

$$= 2a\sin\frac{\phi}{2}.$$

$$S = \int_0^\phi \frac{ds}{d\phi}\, d\phi = 2a \int_0^\phi \sin\frac{\phi}{2}\, d\phi$$

$$= 4a\left(1 - \cos\frac{\phi}{2}\right).$$

$$\frac{ds}{dt} = \dot\phi\, 2a\sin\frac{\phi}{2}$$

K.E. $T = \frac{1}{2} m \dot{s}^2 = \frac{m}{2} 4a^2 \dot\phi^2 \sin^2\frac{\phi}{2} = 2ma^2\dot\phi^2 \sin^2\frac{\phi}{2}$

P.E. $V = mgy = mga(\cos\phi - 1). = -2mga\sin^2\frac{\phi}{2}.$

$$L = T - V$$
$$= 2m\sin^2\frac{\phi}{2}\left[a^2\dot\phi^2 + ga\right]$$

$$\frac{\partial L}{\partial \phi} = 2m \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left[a^2\dot{\phi}^2 + ga\right]$$

$$\frac{\partial L}{\partial \dot{\phi}} = 4m \sin^2\frac{\phi}{2}\, a^2\dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 4ma^2\left\{\ddot{\phi}\sin^2\frac{\phi}{2} + \dot{\phi}^2 \sin\frac{\phi}{2}\cos\frac{\phi}{2}\right\}.$$

Lagrange's eqn:

$$2a^2\ddot{\phi}\sin^2\frac{\phi}{2} + 2a^2\dot{\phi}^2\sin\frac{\phi}{2}\cos\frac{\phi}{2} = \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(a^2\dot{\phi}^2 + ga\right)$$

$$2a^2\ddot{\phi}\sin\frac{\phi}{2} + a^2\dot{\phi}^2\cos\frac{\phi}{2} - ga\cos\frac{\phi}{2} = 0$$

though not explicitly stated, but is suggested in the figure, since the two replicas of the cycloid between which the pendulum is hung are identical, the cycloid must be generated by rotating a circle with constant angular velocity, i.e. $\ddot{\phi} = 0$

So $$a^2\dot{\phi}^2\cos\frac{\phi}{2} - ga\cos\frac{\phi}{2} = 0$$

$$\dot{\phi} = \sqrt{\frac{g}{a}}$$

Since pendulum starts (for example) at $\phi = 0$, go through 2π, then back to $\phi = 0$, this counts as one complete cycle and so

$$\tau_0 = \frac{1}{2}\sqrt{\frac{g}{a}} = \sqrt{\frac{g}{l}} \quad \text{for } l = 4a$$

#

indept of amplitude & oscillations are isochronous.

3-10. The soln for underdamped oscillator is

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta) \qquad \text{where } \omega_1 = \sqrt{\omega_0^2 - \beta^2}.$$

after n periods, amplitude has decreased by e, so

$$e^{-\beta \frac{2\pi}{\omega_1} n} = e^{-1}.$$

$$\beta = \frac{\omega_1}{2\pi n}.$$

therefore

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\omega_1^2}{4\pi^2 n^2}}$$

or

$$\omega_1 = \omega_0 \left(1 + \frac{1}{4\pi^2 n^2}\right)^{-\frac{1}{2}}.$$

$$\simeq \omega_0 \left(1 - \frac{1}{8\pi^2 n^2}\right) \qquad \text{for } n \text{ large}$$

$\#$.

3-17. average KE is

$$\langle T \rangle = \frac{mA^2}{4} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2} \qquad \text{(3.70 in book).}$$

the resonance freq. is $\omega = \omega_0$
and assume now $\omega = 2^n \omega_0$ so that ω is n octaves above ω_0.

then $$\langle T \rangle = \frac{mA^2}{4} \frac{2^{2n}\omega_0^2}{(\omega_0^2 - 2^{2n}\omega_0^2)^2 + 4\, 2^{2n}\omega_0^2\beta^2}$$

$$= \frac{mA^2}{4} \frac{\omega_0^2}{\omega_0^4(2^n - 2^{-n})^2 + 4\omega_0^2\beta^2}$$

$$= \frac{mA^2}{4} \frac{1}{(2^n - 2^{-n})^2 + 4\omega_0^2\beta^2}.$$

But the same expression also holds true for ω being n octaves below ω_0, in which case $n \to -n$ and the above expression remains unchanged, so $\langle T \rangle$ is the same for ω being n octaves above or n octaves below the resonance freq.

3-35. For damped oscillator,

$$G(t,t') = \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin\omega_1(t-t') \; U(t-t')$$

where $U(t-t')$

$$= \begin{cases} 1 & t \geq t' \\ 0 & t < t' \end{cases}$$

is the unit step fcn.

$$F(t) = \begin{cases} 0 & t < 0 \\ ma\sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & t > \frac{\pi}{\omega}. \end{cases}$$

$$= ma\sin\omega t \left[U(t) - U(t - \tfrac{\pi}{\omega}) \right]$$

So in Laplace transform domain,

$$G(s) \equiv \mathcal{L}\{G(t)\}$$

$$= \frac{1}{m\omega_1} \frac{\omega_1}{(s+\beta)^2 + \omega_1^2} = \frac{1}{m}\left[\frac{1}{(s+\beta)^2 + \omega_1^2} \right]$$

$$F(s) \equiv \mathcal{L}\{F(t)\}$$

$$= \mathcal{L}\{ma\sin\omega t\, U(t) - ma\sin\omega t\, U(t - \tfrac{\pi}{\omega})\}$$

$$= ma\,\mathcal{L}\{\sin\omega t\} - ma\, e^{-\frac{\pi}{\omega}s} \mathcal{L}\{\sin\omega(t + \tfrac{\pi}{\omega})\}$$

$$= ma\left\{ \mathcal{L}\{\sin\omega t\} + e^{-\frac{\pi}{\omega}s} \mathcal{L}\{\sin\omega t\} \right\}$$

$$= ma\left\{ \frac{\omega}{s^2+\omega^2} + e^{-\frac{\pi}{\omega}s} \frac{\omega}{s^2+\omega^2} \right\}$$

$$= \frac{ma\omega}{s^2+\omega^2}\left(1+e^{-\frac{\pi}{\omega}s}\right)$$

$$X(s) = G(s)\ F(s)$$

$$= a\omega\left(\frac{1}{s^2+\omega^2}\right)\left[\frac{1}{(s+\beta)^2+\omega_1^2}\right]\left(1+e^{-\frac{\pi}{\omega}s}\right)$$

$$= a\omega\left(1+e^{-\frac{\pi}{\omega}s}\right)\left[\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right)\frac{1}{2i\omega}\right] \times \left[\left(\frac{1}{s+\beta-i\omega_1} - \frac{1}{s+\beta+i\omega_1}\right)\frac{1}{2i\omega_1}\right].$$

$$= \frac{-a\left(1+e^{-\frac{\pi}{\omega}s}\right)}{4\omega_1}\left[\frac{1}{(s-i\omega)(s+\beta-i\omega_1)} - \frac{1}{(s-i\omega)(s+\beta+i\omega_1)} - \frac{1}{(s+i\omega)(s+\beta-i\omega_1)} + \frac{1}{(s+i\omega)(s+\beta+i\omega_1)}\right]$$

since $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \frac{e^{bt}-e^{at}}{b-a}.$

$$\mathcal{L}^{-1}\{e^{-cs}H(s)\} = U(t-c)H(t-c).$$

— you can look these up in Laplace transform tables. If you are not familiar with the use of Laplace transform to solve ODE or IVP, consult, eg. Schaum's outline on Laplace transform for a practical reference.

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$= -\frac{a}{4\omega_1}\Bigg\{ \frac{e^{(-\beta+i\omega_1)t} - e^{i\omega t}}{-\beta+i\omega_1 - i\omega} + \frac{e^{(-\beta+i\omega_1)(t-\frac{\pi}{\omega})} - e^{i\omega(t-\frac{\pi}{\omega})}}{-\beta+i\omega_1-i\omega}$$

$$- \frac{e^{(-\beta-i\omega_1)t} - e^{i\omega t}}{-\beta - i\omega_1 - i\omega} - \frac{e^{(-\beta-i\omega_1)(t-\frac{\pi}{\omega})} - e^{i\omega(t-\frac{\pi}{\omega})}}{-\beta - i\omega_1 - i\omega}$$

$$- \frac{e^{(-\beta+i\omega_1)t} - e^{-i\omega t}}{-\beta+i\omega_1+i\omega} - \frac{e^{(-\beta+i\omega_1)(t-\frac{\pi}{\omega})} - e^{-i\omega(t-\frac{\pi}{\omega})}}{-\beta+i\omega_1+i\omega}$$

$$+ \frac{e^{(-\beta-i\omega_1)t} - e^{-i\omega t}}{-\beta - i\omega_1+i\omega} + \frac{e^{(-\beta-i\omega_1)(t-\frac{\pi}{\omega})} - e^{-i\omega(t-\frac{\pi}{\omega})}}{-\beta-i\omega_1+i\omega}\Bigg\}$$

notice that the above are just 8 pairs of terms of the form

$$\frac{e^{i\alpha t}}{A+iB} + \frac{e^{-i\alpha t}}{A-iB} = \frac{2(A\cos\alpha t + B\sin\alpha t)}{A^2+B^2}.$$

for clarity we let $K(A,B,\theta) \equiv \dfrac{2(A\cos\theta + B\sin\theta)}{A^2+B^2}$.

then the above can be written as

$$X(t) = \frac{-a}{4\omega_1} \times$$

$$\Big\{ e^{-\beta t} K(-\beta, \omega_1 - \omega, \omega_1 t) + e^{-\beta(t-\frac{\pi}{\omega})} K(-\beta, \omega_1 - \omega, \omega_1 (t - \tfrac{\pi}{\omega}))$$

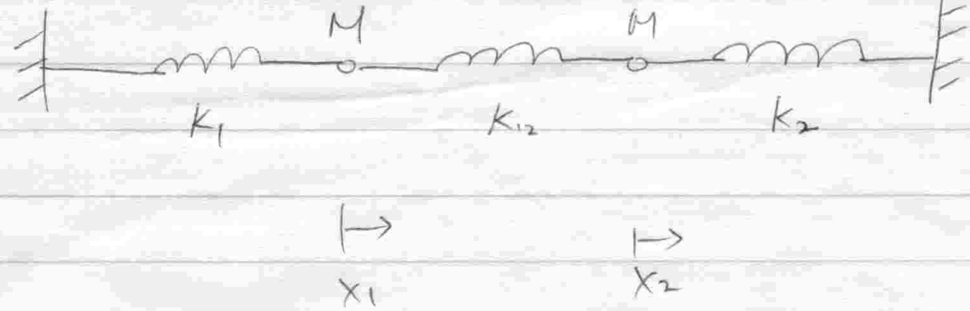
$$- K(-\beta, \omega_1 - \omega, \omega t) - K(-\beta, \omega_1 - \omega, \omega (t - \tfrac{\pi}{\omega}))$$

$$- e^{-\beta t} K(-\beta, \omega_1 + \omega, \omega_1 t) - e^{-\beta(t-\frac{\pi}{\omega})} K(-\beta, \omega_1 + \omega, \omega_1 (t - \tfrac{\pi}{\omega}))$$

$$+ K(-\beta, \omega_1 + \omega, \omega t) + K(-\beta, \omega_1 + \omega, \omega (t - \tfrac{\pi}{\omega})) \Big\}$$

#

12-1.



$$M\ddot{x}_1 = -k_1 x_1 + k_{12}(x_2 - x_1) = -(k_1 + k_{12})x_1 + k_{12}x_2 .$$
$$M\ddot{x}_2 = -k_2 x_2 - k_{12}(x_2 - x_1) = k_{12}x_1 - (k_2 + k_{12})x_2 .$$

$$M\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}-(k_1+k_{12}) & k_{12}\\ k_{12} & -(k_2+k_{12})\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

now assume $x_1 \sim e^{-i\omega t}$, $x_2 \sim e^{-i\omega t}$

$$\begin{bmatrix}-(k_1+k_{12})+\omega^2 M & k_{12}\\ k_{12} & -(k_2+k_{12})+\omega^2 M\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}\omega^2 M-(k_1+k_{12}) & k_{12}\\ k_{12} & \omega^2 M-(k_2+k_{12})\end{vmatrix} = 0$$

$$M^2\omega^4 - M(k_1+k_2+2k_{12})\omega^2 + \left[k_1k_2 + (k_1+k_2)k_{12}\right] = 0 .$$

$$\omega^2 = \frac{M(k_1+k_2+2k_{12}) \pm \sqrt{M^2(k_1+k_2+2k_{12})^2 - 4M^2\left[k_1k_2+(k_1+k_2)k_{12}\right.}}{2M^2}$$

$$\omega_{1,2} = \left[\frac{(k_1+k_2+2k_{12}) \pm \sqrt{(k_1-k_2)^2 + 4k_{12}^2}}{2M}\right]^{\frac{1}{2}} \text{ — normal freq.}$$

#

no coupling $\Rightarrow k_{12} = 0$

natural freq. of springs are

$$\omega_1' = \sqrt{\frac{k_1}{M}} \quad , \quad \omega_2' = \sqrt{\frac{k_2}{M}}.$$

$$\omega_1 \gtreqless \omega_1'$$

$$k_1 + k_2 + 2k_{12} \pm \sqrt{(k_1 - k_2)^2 + 4k_{12}^2} \gtreqless 2k_1.$$

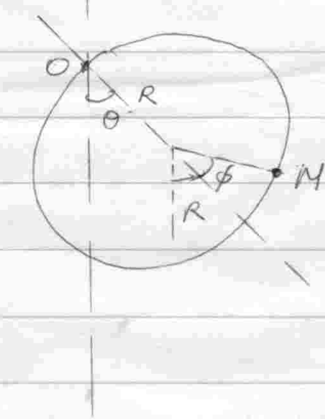
$$(k_1 - k_2)^2 + 4k_{12}^2 \gtreqless (k_1 - k_2 - 2k_{12})^2.$$

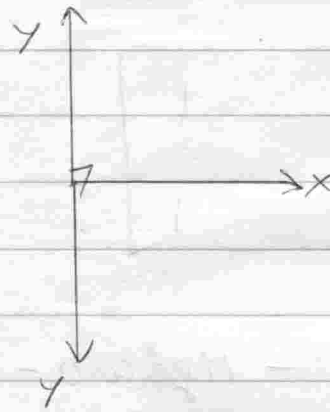
$$= (k_1 - k_2)^2 + 4k_{12}^2 - 4k_{12}(k_1 - k_2).$$

$$\Rightarrow \quad \omega_1 > \omega_1'$$

similarly $\quad \omega_2 < \omega_2'.$

12-16.





position of mass M =

$$x = R\sin\phi + R\sin\theta$$
$$y = -R\cos\phi - R\cos\theta$$

$$\dot{x}^2+\dot{y}^2 = R^2\left[\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi\sin\theta+2\dot{\phi}\dot{\theta}\cos\phi\cos\theta\right]$$
$$= R^2\left[\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\cos(\theta-\phi)\right]$$

assume small oscillations, $\cos(\theta-\phi) \simeq 1$

$$\dot{x}^2+\dot{y}^2 = R^2\left[\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\right]$$

moment of inertia of hoop about centre $I = MR^2$

about O the hinging pt, $I_o = MR^2+MR^2$
$$= 2MR^2$$

KE $\quad T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I_o\dot{\theta}^2$

$$= \frac{1}{2}MR^2(\dot{\phi}^2+3\dot{\theta}^2+2\dot{\theta}\dot{\phi})$$

$\underline{M}$ is given by

$$\underline{M} = MR^2 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

PE $\quad V = -MgR\cos\theta - MgR(\cos\theta + \cos\phi).$

$$= -MgR(2\cos\theta + \cos\phi).$$

$$\simeq -MgR\left[2\left(1 - \frac{\theta^2}{2}\right) + 1 - \frac{\phi^2}{2}\right]$$

$$= -MgR\left(3 - \theta^2 - \frac{\phi^2}{2}\right).$$

$$= MgR\left(\theta^2 + \frac{\phi^2}{2}\right)$$ leaving out the unimportant constant term.

$$\underline{V} = MgR \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det\left\{\begin{bmatrix} 2MgR & 0 \\ 0 & MgR \end{bmatrix} - \omega^2 \begin{bmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{bmatrix}\right\} = 0$$

$$\begin{vmatrix} 2MgR - 3MR^2\omega^2 & -MR^2\omega^2 \\ -MR^2\omega^2 & MgR - MR^2\omega^2 \end{vmatrix} = 0$$

$$3(MR^2)^2\omega^4 - 5M^2R^3 g\omega^2 + 2(MgR)^2 - (MR^2\omega^2)^2 = 0.$$

$$2(MR^2)^2\omega^4 - 5(MR^2)(MgR)\omega^2 + 2(MgR)^2 = 0$$

$$\cancel{[2(MR^2)\omega^2 - 1][(MR^2)\omega^2}$$

$$[2(MR^2)\omega^2 - MgR][(MR^2)\omega^2 - 2(MgR)] = 0$$

$$\omega_2 = \sqrt{\frac{g}{2R}} \quad , \quad \omega_1 = \sqrt{\frac{2g}{R}}.$$

for $\omega_1 = \sqrt{\frac{2g}{R}}$,

$$\left[\begin{array}{cc|c} 2MgR - 3MR^2\omega_1^2 & -MR^2\omega_1^2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} -4MgR & -2MgR & 0 \end{array}\right].$$

$\therefore$ eigenvector $|\omega_1\rangle = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

for $\omega_2 = \sqrt{\frac{g}{2R}}$

$$\left[\begin{array}{cc|c} -MR^2\omega_2^2 & MgR - MR^2\omega_2^2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} -\frac{MgR}{2} & \frac{MgR}{2} & 0 \end{array}\right].$$

$\therefore$ eigenvector $|\omega_2\rangle = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$

thus initial cond. for NM 1

$2\theta = \phi$ at $t=0$

initial cond for NM 2

$\theta = -\phi$ at $t=0$.

12-21. KE

$$T = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} M \dot{x}_2^2 + \frac{1}{2} M \dot{x}_3^2$$

$$\therefore \quad \underline{M} = \begin{bmatrix} M & & \\ & M & \\ & & M \end{bmatrix}$$

PE

$$V = \frac{1}{2} \left[k_1 (x_1^2 + x_3^2) + k_2 x_2^2 + k_3 (x_1 x_2 + x_2 x_3) \right]$$

$$\underline{V} = \begin{bmatrix} k_1 & k_3/2 & 0 \\ k_3/2 & k_2 & k_3/2 \\ 0 & k_3/2 & k_1 \end{bmatrix}$$

$$\left| \underline{V} - \omega^2 \underline{M} \right| = 0$$

$$\begin{vmatrix} k_1 - \omega^2 M & k_3/2 & 0 \\ k_3/2 & k_2 - \omega^2 M & k_3/2 \\ 0 & k_3/2 & k_1 - \omega^2 M \end{vmatrix} = 0.$$

$$(k_1 - \omega^2 M) \left[\omega^4 M^2 - \omega^2 M (k_1 + k_2) \right] = 0.$$

$$\omega_1 = \sqrt{\frac{k_1}{M}}, \quad \omega_2 = \sqrt{\frac{k_1 + k_2}{M}}, \quad \omega_3 = 0.$$

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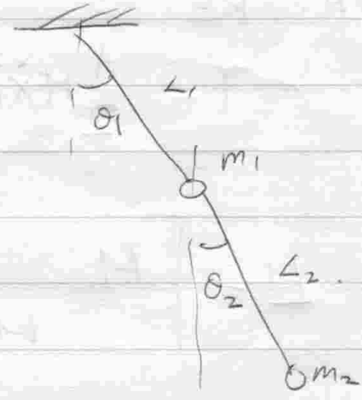
For zero eigenfreq., the entire system comprising the three oscillators undergoes translational motion, there is no vibration within the system.

12-27.



$$x_1 = L_1 \sin\theta_1$$
$$y_1 = L_1 \cos\theta_1.$$

$$x_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$$
$$y_2 = L_1 \cos\theta_1 + L_2 \cos\theta_2.$$

$$\dot{x}_1^2 + \dot{y}_1^2 = L_1^2 \dot{\theta}_1^2$$

$\simeq 1$ small amplitude assumptio

$$\dot{x}_2^2 + \dot{y}_2^2 = L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$T = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \right)$$

$$V = - m_1 g y_1 - m_2 g y_2$$
$$= - m_1 g L_1 \cos\theta_1 - m_2 g (L_1 \cos\theta_1 + L_2 \cos\theta_2).$$

small amplitude assumption $\Rightarrow \cos\theta \approx 1 - \frac{\theta^2}{2}$

$$V = + m_1 g L_1 \frac{\theta^2}{2} + \frac{m_2 g}{2} \left(L_1 \theta_1^2 + L_2 \theta_2^2 \right)$$

dropping the unimportant constant terms

$$\therefore \underline{M} = \begin{bmatrix} m_1 L_1^2 + m_2 L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}$$

$$\underline{V} = \begin{bmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}.$$

$$|\underline{V} - \underline{M}\omega^2| = 0$$

$$\begin{vmatrix} (m_1+m_2)gL_1 - (m_1+m_2)L_1^2\omega^2 & -m_2L_1L_2\omega^2 \\ -m_2L_1L_2\omega^2 & m_2gL_2 - m_2L_2^2\omega^2 \end{vmatrix} = 0$$

$$(m_1+m_2)m_2(L_1L_2)^2\omega^4 - (m_1+m_2)m_2L_1L_2(L_1+L_2)g\omega^2 + (m_1+m_2)m_2g^2L_1L_2$$
$$- m_2^2(L_1L_2)^2\omega^4 = 0$$

$$m_1m_2(L_1L_2)^2\omega^4 - (m_1+m_2)m_2L_1L_2(L_1+L_2)g\omega^2 + (m_1+m_2)m_2g^2L_1L_2 = 0$$

$$m_1(L_1L_2)\omega^4 - (m_1+m_2)(L_1+L_2)g\omega^2 + (m_1+m_2)g^2 = 0.$$

$$\omega_{1,2} = \left[\frac{g(L_1+L_2)(m_1+m_2) \pm \sqrt{g^2(m_1+m_2)[m_1(L_1-L_2)^2 + m_2(L_1+L_2)^2]}}{2L_1L_2m_1}\right]^{\frac{1}{2}}.$$

— normal freq.

$$\cancel{\alpha(m_1+m_2)gL_1 - \beta(m_1+m_2)L_1^2\omega_{1,2}^2 = m_2L_1L_2\omega_{1,2}^2}.$$

#.

$$\alpha[(m_1+m_2)gL_1 - (m_1+m_2)L_1^2\omega_{1,2}^2] = \beta[m_2L_1L_2\omega_{1,2}^2]$$

$$|\omega_{1,2}\rangle = \begin{bmatrix} m_2L_1L_2\omega_{1,2}^2 \\ (m_1+m_2)gL_1 - (m_1+m_2)L_1^2\omega_{1,2}^2 \end{bmatrix}$$ — normal modes.

#

In[11]:= (* 1a. *)

```
(* one-parameter variational solution *)
```

In[167]:=

```
x1[t_] := v0 t + a t^3;
f[x1] = -m w^2 x1[t];
L = Simplify[∫_0^(Pi/w) (x1''[t] - f[x1] / m)^2 dt]
```

Out[169]= $\frac{\pi^3 \left(3 a^2 \left(420 + 84 \pi^2 + 5 \pi^4\right) + 42 a \left(10 + \pi^2\right) v0\, w^2 + 35\, v0^2 w^4\right)}{105\, w^3}$

In[21]:= `ans1 = Solve[∂_a L == 0, a][[1, 1]]`

Out[21]= $a \to -\frac{7 \left(10 + \pi^2\right) v0\, w^2}{420 + 84 \pi^2 + 5 \pi^4}$

```
(* 1b. *)

(* two-parameter variational solution *)

x2[t_] := v0 t + a t^3 + b t^5;
f[x2] = -m w^2 x2[t];
L = Simplify[∫_0^(Pi/w) (x2''[t] - f[x2] / m)^2 dt]
```

Out[174]= $\frac{1}{3465\, w^7} \left(\pi^3 \left(315 b^2 \pi^8 + 770 b \pi^6 \left(20 b + a w^2\right) + 1386 \pi^2 w^2 \left(20 b + a w^2\right) \left(6 a + v0\, w^2\right) + 1155 w^4 \left(6 a + v0\, w^2\right)^2 + 495 \pi^4 \left(400 b^2 + 52 a b w^2 + a^2 w^4 + 2 b\, v0\, w^4\right)\right)\right)$

In[25]:= `ans2 = Solve[{∂_a L == 0, ∂_b L == 0}, {a, b}][[1, {1, 2}]]`

Out[25]= $\left\{a \to -\frac{18 \left(55440\, v0\, w^2 + 6160 \pi^2 v0\, w^2 + 290 \pi^4 v0\, w^2 + 7 \pi^6 v0\, w^2\right)}{5987520 + 665280 \pi^2 + 43200 \pi^4 + 1512 \pi^6 + 35 \pi^8},\right.$
$\left. b \to \frac{99 \left(504\, v0 + 28 \pi^2 v0 + \pi^4 v0\right) w^4}{5987520 + 665280 \pi^2 + 43200 \pi^4 + 1512 \pi^6 + 35 \pi^8}\right\}$

(* 2 *)

In[170]:= $$\mathbf{x1[t]\ /.\ \left\{a \to -\frac{7\,(10+\pi^2)\,v0\,w^2}{420+84\,\pi^2+5\,\pi^4}\right\}\ /.\ \{w \to 1,\ v0 \to 1\}}$$

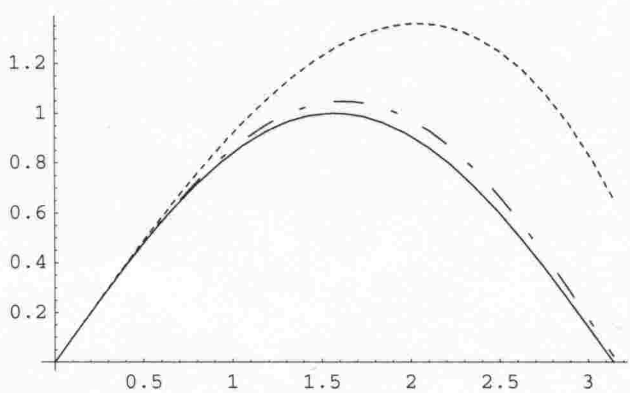
Out[170]= $$t - \frac{7\,(10+\pi^2)\,t^3}{420+84\,\pi^2+5\,\pi^4}$$

In[177]:= $$\mathbf{x2[t]\ /.\ \left\{a \to -\frac{18\,(55440\,v0\,w^2+6160\,\pi^2\,v0\,w^2+290\,\pi^4\,v0\,w^2+7\,\pi^6\,v0\,w^2)}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8},\right.}$$
$$\mathbf{\left.b \to \frac{99\,(504\,v0+28\,\pi^2\,v0+\pi^4\,v0)\,w^4}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8}\right\}\ /.\ \{w \to 1,\ v0 \to 1\}}$$

Out[177]= $$t - \frac{18\,(55440+6160\,\pi^2+290\,\pi^4+7\,\pi^6)\,t^3}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8} + \frac{99\,(504+28\,\pi^2+\pi^4)\,t^5}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8}$$

In[178]:= $$\mathbf{Plot\left[\left\{\left\{t-\frac{7\,(10+\pi^2)\,t^3}{420+84\,\pi^2+5\,\pi^4}\right\},\ \left\{t-\frac{18\,(55440+6160\,\pi^2+290\,\pi^4+7\,\pi^6)\,t^3}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8}+\frac{99\,(504+28\,\pi^2+\pi^4)\,t^5}{5987520+665280\,\pi^2+43200\,\pi^4+1512\,\pi^6+35\,\pi^8}\right\},\right.\right.}$$

```
     Sin[t]},
    {t, 0, Pi}, PlotStyle ->
     {{Dashing[{0.01, 0.01, 0.01, 0.01}]}, {Dashing[{0.01, 0.05, 0.05, 0.05}]}, {}}]
```



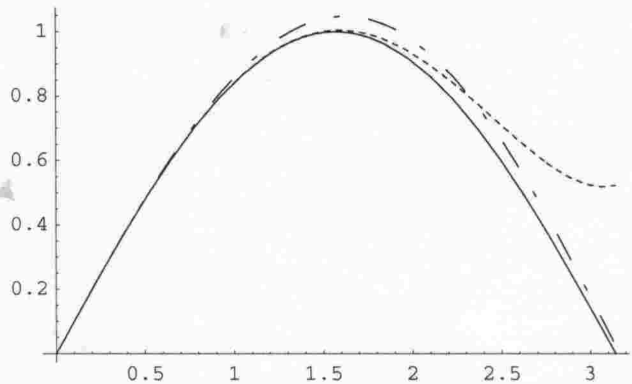
Legend : Dotted line : one - parameter solution
Dot - Dashed line : two - parameter solution
Solid line : Exact solution

It is seen that the two - parameter variational solution fits more closely to the exact solution, as expected.

In[179]:= (* 3 *)

$$\text{Plot}\Big[\Big\{\Big\{t - \frac{t^3}{6} + \frac{t^5}{120}\Big\}, \Big\{t - \frac{18\,(55440 + 6160\,\pi^2 + 290\,\pi^4 + 7\,\pi^6)\,t^3}{5987520 + 665280\,\pi^2 + 43200\,\pi^4 + 1512\,\pi^6 + 35\,\pi^8} + \frac{99\,(504 + 28\,\pi^2 + \pi^4)\,t^5}{5987520 + 665280\,\pi^2 + 43200\,\pi^4 + 1512\,\pi^6 + 35\,\pi^8}\Big\}, \text{Sin}[t]\Big\},$$

```
  {t, 0, Pi}, PlotStyle →
   {{Dashing[{0.01, 0.01, 0.01, 0.01}]}, {Dashing[{0.01, 0.05, 0.05, 0.05}]}, {}}]
```



Legend : Dotted line : Taylor series solution
Dot - Dashed line : two - parameter solution
Solid line : Exact solution

In this case, the two - parameter solution works better than the Taylor series solution. It is because Taylor series is a small argument expansion and as such works good for small argument only. Thus it works better at small t. As can be seen, as t becomes larger, it deviates more from the exact solution. On the other hand, the variational solution works better over range, as it is designed to do so.

```
(* 4 *)

(* one-parameter variational solution *)
```

In[216]:=
```
x1[t_] := v0 t + a t ^ 3;
f[x1] = -m w ^ 2 x1[t];
```
$$L = \text{Simplify}\left[\int_0^{Pi/(2 w)} (x1''[t] - f[x1] / m) \wedge 2\, dt\right]$$

Out[218]=
$$\frac{\pi^3 \left(15 a^2 \pi^4 + 168 a \pi^2 (6 a + v0 w^2) + 560 (6 a + v0 w^2)^2\right)}{13440 w^3}$$

In[219]:=
$$\text{ans1} = \text{Simplify}[\text{Solve}[\partial_a L == 0, a][[1, 1]]]$$

Out[219]=
$$a \to -\frac{28 (40 + \pi^2) v0 w^2}{6720 + 336 \pi^2 + 5 \pi^4}$$

```
(* two-parameter variational solution *)
```

In[220]:=
```
x2[t_] := v0 t + a t ^ 3 + b t ^ 5;
f[x2] = -m w ^ 2 x2[t];
```
$$L = \text{Simplify}\left[\int_0^{Pi/(2 w)} (x2''[t] - f[x2] / m) \wedge 2\, dt\right]$$

Out[222]=
$$\frac{1}{7096320 w^7} \left(\pi^3 \left(315 b^2 \pi^8 + 3080 b \pi^6 (20 b + a w^2) + 88704 \pi^2 w^2 (20 b + a w^2) (6 a + v0 w^2) + 295680 w^4 (6 a + v0 w^2)^2 + 7920 \pi^4 (400 b^2 + 52 a b w^2 + a^2 w^4 + 2 b v0 w^4)\right)\right)$$

In[223]:=
$$\text{ans2} = \text{Solve}[\{\partial_a L == 0,\ \partial_b L == 0\}, \{a, b\}][[1, \{1, 2\}]]$$

Out[223]=
$$\left\{a \to -\frac{72 (3548160 v0 w^2 + 98560 \pi^2 v0 w^2 + 1160 \pi^4 v0 w^2 + 7 \pi^6 v0 w^2)}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8},\ b \to \frac{1584 (8064 v0 + 112 \pi^2 v0 + \pi^4 v0) w^4}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8}\right\}$$

In[224]:=
$$x1[t]\ /.\ \left\{a \to -\frac{28 (40 + \pi^2) v0 w^2}{6720 + 336 \pi^2 + 5 \pi^4}\right\}\ /.\ \{w \to 1,\ v0 \to 1\}$$

Out[224]=
$$t - \frac{28 (40 + \pi^2) t^3}{6720 + 336 \pi^2 + 5 \pi^4}$$

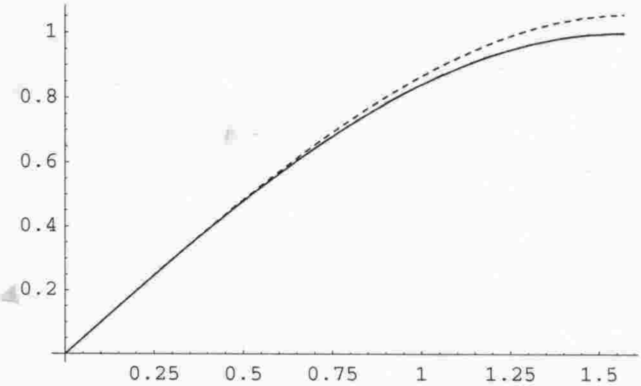
In[225]:=
$$x2[t]\ /.\ \left\{a \to -\frac{72 (3548160 v0 w^2 + 98560 \pi^2 v0 w^2 + 1160 \pi^4 v0 w^2 + 7 \pi^6 v0 w^2)}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8},\ b \to \frac{1584 (8064 v0 + 112 \pi^2 v0 + \pi^4 v0) w^4}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8}\right\}\ /.\ \{w \to 1,\ v0 \to 1\}$$

Out[225]=
$$t - \frac{72 (3548160 + 98560 \pi^2 + 1160 \pi^4 + 7 \pi^6) t^3}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8} + \frac{1584 (8064 + 112 \pi^2 + \pi^4) t^5}{1532805120 + 42577920 \pi^2 + 691200 \pi^4 + 6048 \pi^6 + 35 \pi^8}$$

In[227]:=

$$\texttt{Plot}\left[\left\{\left\{t-\frac{28\,(40+\pi^2)\,t^3}{6720+336\,\pi^2+5\,\pi^4}\right\},\left\{t-\frac{72\,(3548160+98560\,\pi^2+1160\,\pi^4+7\,\pi^6)\,t^3}{1532805120+42577920\,\pi^2+691200\,\pi^4+6048\,\pi^6+35\,\pi^8}+\frac{1584\,(8064+112\,\pi^2+\pi^4)\,t^5}{1532805120+42577920\,\pi^2+691200\,\pi^4+6048\,\pi^6+35\,\pi^8}\right\},\texttt{Sin}[t]\right\},\right.$$

```
  {t, 0, Pi / 2}, PlotStyle →
   {{Dashing[{0.01, 0.01, 0.01, 0.01}]}, {Dashing[{0.01, 0.05, 0.05, 0.05}]}, {}}]
```



Legend : Dotted line : one - parameter solution

Dot - Dashed line : two - parameter solution

Solid line : Exact solution

Obviously, as one narrows down the range to do the variational calculation, one should be able to obtain better approximations to the exact solution, as is illustrated on this graph.

soln 11.

1. $$\ddot{x} = -\omega_0^2 x - 2\beta\dot{x}$$

so $f(x,\dot{x}) = -2\beta\dot{x}$

$$\dot{A} = -\frac{1}{\omega_0}\langle f \sin[\omega_0 t + \phi(t)]\rangle$$

$$= \frac{2\beta}{\omega_0}\left(\frac{\omega_0}{2\pi}\right)\int_0^{\frac{2\pi}{\omega_0}} dt \; \dot{x}\sin[\omega_0 t + \phi(t)].$$

with $x = A(t)\cos[\omega_0 t + \phi(t)]$

$$\dot{x} = \dot{A}\cos[\omega_0 t + \phi(t)] - A(\omega_0 + \dot{\phi})\sin[\omega_0 t + \phi(t)]$$

$$\int_0^{\frac{2\pi}{\omega_0}} dt \left\{\dot{A}\cos[\omega_0 t + \phi]\sin[\omega_0 t + \phi] - A(\omega_0 + \dot{\phi})\sin^2[\omega_0 t + \phi]\right\}.$$

$$\dot{A} = \frac{\beta}{\pi}\int_0^{\frac{2\pi}{\omega_0}} dt \left\{\dot{A}\sin[\omega_0 t + \phi]\cos[\omega_0 t + \phi] - A(\omega_0 + \dot{\phi})\sin^2[\omega_0 t + \phi]\right\}$$

$$= \frac{\omega_0\beta}{\pi}\int_0^{\frac{2\pi}{\omega_0}} dt \left\{\underbrace{\cancel{\frac{\dot{A}}{A\omega_0}}}_{0}\sin[\omega_0 t + \phi]\cos[\omega_0 t + \phi] - \left(1 + \underbrace{\cancel{\frac{\dot{\phi}}{\omega_0}}}_{0}\right)\sin^2[\omega_0 t + \phi]\right\} A$$

since damping is small, the characteristic decay time given by $\frac{A}{\dot{A}} \ll \frac{1}{\omega_0}$, also ϕ is approximately const, $\dot{\phi} \ll \omega_0$.

then

$$\dot{A} \simeq -\beta A$$

$\therefore \quad A(t) = A_0 e^{-\beta t}$

Similarly

$$\dot{\phi} = \frac{2\beta}{A\omega_0}\left(\frac{\omega_0}{2\pi}\right)\int_0^{\frac{2\pi}{\omega_0}} dt \cdot \dot{x} \cos[\omega_0 t + \phi]$$

$$= \frac{\beta}{\pi A}\int_0^{\frac{2\pi}{\omega_0}} dt \left\{ \dot{A}\cos^2[\omega_0 t + \phi] - A(\omega_0 + \dot{\phi})\sin[\omega_0 t + \phi]\cos[\omega_0 t + \phi] \right\}$$

$$\simeq \frac{\beta\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} dt \left\{ \cancelto{0}{\frac{\dot{A}}{A}} \cos^2[\omega_0 t + \phi] - (1 + \cancelto{0}{\frac{\dot{\phi}}{A}}) \sin[\omega_0 t + \phi]\cos[\omega_0 t + \phi] \right\}$$

$$= 0.$$

$\therefore \quad \phi = -\alpha = \text{Const}.$

$\therefore \quad x(t) = A_0 e^{-\beta t} \cos(\omega_0 t - \alpha).$

#

2. The exact solution is

$$x(t) = A_0 e^{-\beta t} \cos(\omega_1 t - \alpha)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

Expanding $X(t)$ up to 1^{st} order in β, notice the first order term is

$$A_0 e^{-\beta t} \frac{d}{d\beta} \cos(\omega_1 t - \alpha) \Big|_{\beta=0}$$

$$= + A_0 e^{-\beta t} \sin(\omega_1 t - \alpha) \frac{2\beta}{\sqrt{\omega_0^2 - \beta^2}} \Big|_{\beta=0} = 0.$$

hence the 1^{st} order correction in β ~~is~~ actually vanishes, and the 0^{th} order term is

and $x(t) \simeq A_0 e^{-\beta t} \cos(\omega_0 t - \alpha)$

which agrees with Q1.

4-3. $V(x) = -\frac{\lambda}{3}x^3$,

Hamiltonian $H = \frac{p^2}{2m} - \frac{\lambda}{3}x^3$

So the trajectory in phase space is given by

$$p = \pm\sqrt{2mE + \frac{2m\lambda}{3}x^3}$$ where $E = H$ is the energy

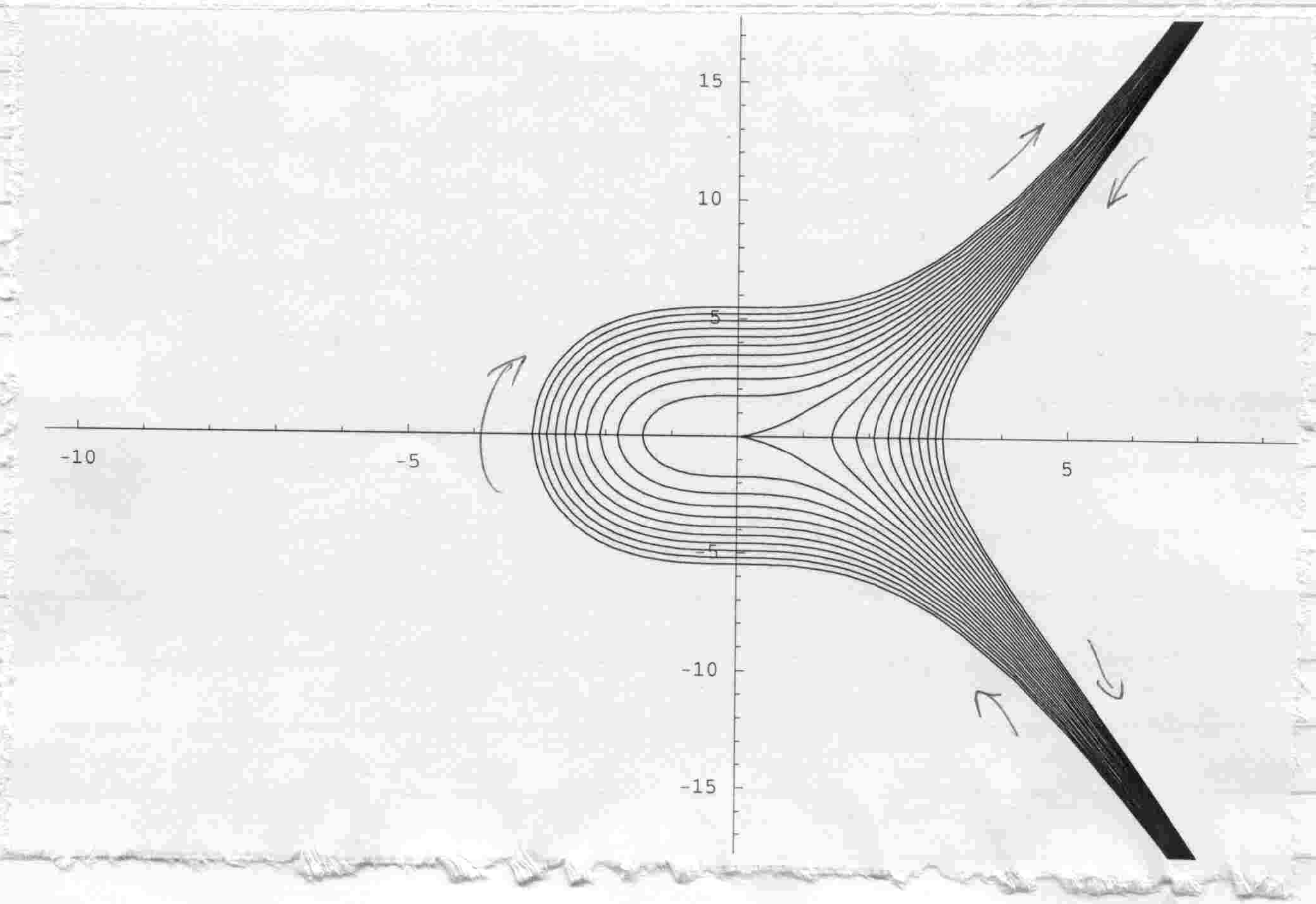
The phase diagram is plotted below. From Hamilton's eqns

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = \lambda x^2$$

$$\Rightarrow \frac{dp}{dx} = \frac{\lambda m x^2}{p}$$

we see the direction of flow is as indicated in the figure.


15
10
5
-10
-5
5
-5
-10
-15

4-4. $\ddot{x} - (a - b\dot{x}^2)\dot{x} + \omega_0^2 x = 0$

differentiate w.r.t. t :

$$\dddot{x} - (a - 3b\dot{x}^2)\ddot{x} + \omega_0^2 \dot{x} = 0$$

Sub. $\dot{x} = \frac{y}{y_0}\sqrt{\frac{a}{3b}}$

$$\frac{\ddot{y}}{y_0}\sqrt{\frac{a}{3b}} - \left[a - 3b\left(\frac{y}{y_0}\right)^2 \frac{a}{3b}\right]\frac{\dot{y}}{y_0}\sqrt{\frac{a}{3b}} + \omega_0^2 \frac{y}{y_0}\sqrt{\frac{a}{3b}} = 0$$

$$\ddot{y} - \frac{a}{y_0^2}(y_0^2 - y^2)\dot{y} + \omega_0^2 y = 0$$

#.

4-8. $V(x) = \begin{cases} F_0 x & , x > 0 \\ -F_0 x & x < 0 \end{cases}$

Hamiltonian

$$H = \frac{p^2}{2m} \pm F_0 x \qquad \begin{pmatrix} x>0 \\ x<0 \end{pmatrix}$$

$$p = \pm\sqrt{2mE \mp 2F_0 m x} \qquad \text{— phase trajectory}$$

Hamilton's eqns

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

$$\dot{p} = -\frac{\partial H}{\partial x} = \mp F_0$$

$$\Rightarrow \frac{dp}{dx} = \mp\frac{mF_0}{p} \qquad \text{— indicating the direction of trajectory}$$

eqn of motion is

$$\ddot{x} = \mp\frac{F_0}{m}.$$

due to symmetry only need to consider a quarter of a whole period, then

$$\ddot{x} = -\frac{F_0}{m} \qquad \text{with} \quad \dot{x}(0) = 0, \; x(0) = A > 0$$

$$x = -\frac{F_0}{2m}t^2 + A$$

putting $x=0$ get a quarter of period $= \sqrt{\frac{2mA}{F_0}}$

so total period $T = 4\sqrt{\frac{2mA}{F_0}}$

#.