PHYS 410

HW1

1a. $ds = \sqrt{dx^2 + dy^2 + dz^2}$

$$= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}$$

$$\frac{dz}{dx} = 2(ax + ayy')$$

i) $S = \int_1^2 ds = \int_1^2 dx\sqrt{1 + (y')^2 + 4a^2(x + yy')^2}$ #

1.b. $f = \sqrt{1+(y')^2 + 4a^2(x'+yy')^2}$

$$f^2 = 1+(y')^2+4a^2(x'+yy')^2 = f^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2f}\left[4a^2 \cdot 2 \cdot (x'+yy')\right]y'$$

$$= \frac{4a^2(x+yy')y'}{f}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2f}\left[2y' + 8a^2(x+yy')y\right] = \frac{y'+4a^2y(x+yy')}{f}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{f\left[y''+4a^2y'(x+yy')+4a^2y(1+yy''+(y'^2))\right] - \left[y'+4a^2y(x+yy')\right]\frac{df}{dx}}{f^2}$$

$$\frac{df}{dx} = \frac{1}{2f}\left[2y'y''+8a^2(x+yy')(1+(y')^2+yy'')\right]$$

$$= \frac{\left[1+(y')^2+4a^2(x+yy')^2\right]\left[y''+4a^2y'(x+yy')+4a^2y(1+(y')^2+yy'')\right] - \left[y'+4a^2y(x+yy')\right]\left[y'y''+4a^2(x+yy')(1+(y')^2+yy'')\right]}{f^3}$$

numerator A

$$\begin{aligned}
= {} & y''+4a^2y'(\cancel{x}+yy')+4a^2y(1+(y')^2+yy'') \\
& +(y')^2y''+4a^2(y')^3(x+yy')+4a^2y(y')^2(1+(y')^2+yy'') \\
& +4a^2y''(x+yy')^2+(4a^2)^2y'(x+yy')^3+(4a^2)^2y(x+yy')^2(1+(y')^2+yy'') \\
& -(y')^2y''-4a^2y'(x+yy')(1+(y')^2+yy'') \\
& -4a^2yy'y''(x+yy')-(4a^2)^2y(x+yy')^2(1+(y')^2+yy'')
\end{aligned}$$

$$\begin{aligned}
= {} & y''+4a^2y(y')^2+4a^2y(1+(y')^2+yy'')+4a^2(y')^3(x+yy')+4a^2y''(x+yy')^2 \\
& +(4a^2)^2y'(x+yy')^3-4a^2yy'y''(x+yy')-4a^2xy'(1+(y')^2+yy'')
\end{aligned}$$

$$= y'' + 4a^2 y (y')^2 + 4a^2 y (1 + (y')^2 + yy')$$

$$+ 4a^2 (x + yy') \left[(y')^3 + y''(x+yy') - yy'y'' \right] + (4a^2)^2 y'(x+yy')^3$$
$$- 4a^2 xy' \left[(y')^2 + yy'' \right].$$

$$A = y'' + 4a^2 \left[y(y')^2 + y(1 + (y')^2 + yy'') + (x+yy')((y')^3 + xy'') \right]$$

$$+ (4a^2)^2 y'(x+yy')^3 - 4a^2 xy' \left[(y')^2 + yy'' \right].$$

$\therefore$ Euler-Lagrange Eqn.

$$A = 4a^2 f^2 (x + yy') y'$$
$$= 4a^2 \left[1 + (y')^2 + 4a^2 (x+yy')^2 \right] (x+yy') y'$$

$$y'' + 4a^2 \left[y(y')^2 + y(1 + (y')^2 + yy'') + (x+yy')(xy'' - y') \right]$$
$$- 4a^2 xy' \left[(y')^2 + yy'' \right] = 0$$

$$(1 + 4a^2 x^2 + 4a^2 y^2)\, y'' + 4a^2 (y - xy')(1 + (y')^2) = 0$$

2. $z = a\rho^2$

$$ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2}$$

$$= d\phi \sqrt{\rho^2 + \left(\frac{d\rho}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2}$$

$$= d\phi \sqrt{\rho^2 + \left(\frac{dz}{d\rho}\right)^2\left(\frac{d\rho}{d\phi}\right)^2 + \left(\frac{d\rho}{d\phi}\right)^2}$$

$$= d\phi \sqrt{\rho^2 + \left(\frac{d\rho}{d\phi}\right)^2 (1 + 4a^2\rho^2)}$$

$$f = \sqrt{\rho^2 + \left(\frac{d\rho}{d\phi}\right)^2 (1 + 4\rho^2 a^2)} = \sqrt{\rho^2 + (\rho')^2 (1 + 4\rho^2 a^2)}$$

$$\frac{\partial f}{\partial \rho} = \frac{1}{2f}\left[2\rho + (\rho')^2 (8a^2\rho)\right] = \frac{\rho + 4a^2\rho(\rho')^2}{f}$$

$$\frac{\partial f}{\partial \rho'} = \frac{1}{2f}\left[2\rho'(1 + 4a^2\rho^2)\right] = \frac{\rho'(1 + 4a^2\rho^2)}{f}$$

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial \rho'}\right) = \frac{f\left[\rho''(1 + 4a^2\rho^2) + \rho'(4a^2 \cdot 2\rho\rho')\right] - \rho'(1 + 4a^2\rho^2)\frac{df}{d\phi}}{f^2}$$

$$\frac{df}{d\phi} = \frac{1}{2f}\left[2\rho\rho' + 2\rho'\rho''(1 + 4a^2\rho^2) + (\rho')^2(8a^2\rho\rho')\right]$$

$$= \frac{\rho\rho' + \rho'\rho''(1 + 4a^2\rho^2) + 4a^2\rho(\rho')^3}{f}$$

$A f_\rho^3 =$

$$\left[\frac{d}{d\phi}\left(\frac{\partial f}{\partial \rho'}\right)\right] f^3$$

$$= \left[\rho^2 + (\rho')^2(1+4a^2\rho^2)\right]\left[\rho''(1+4a^2\rho^2) + \rho'(8a^2\rho\rho')\right]$$

$$- \rho'(1+4a^2\rho^2)\left[\rho\rho' + \rho'\rho''(1+4a^2\rho^2) + 4a^2\rho(\rho')^3\right]$$

$$= \rho^2\rho''(1+4a^2\rho^2) + 8a^2\rho^3(\rho')^2 + \cancel{(\rho')^2\rho''(1+4a^2\rho^2)^2} + 8a^2\rho(\rho')^4(1+4a^2\rho^2)$$

$$- \rho(\rho')^2(1+4a^2\rho^2) - \cancel{(\rho')^2\rho''(1+4a^2\rho^2)^2} - 4a^2\rho(\rho')^4(1+4a^2\rho^2)$$

$$= (1+4a^2\rho^2)\left(\rho^2\rho'' - \rho(\rho')^2 + 4a^2\rho(\rho')^4\right) + 8a^2\rho^3(\rho')^2$$

Euler-Lagrange eqn.

$$A = \left[\rho + 4a^2\rho(\rho')^2\right]\left[\rho^2 + (\rho')^2(1+4a^2\rho^2)\right]$$

$$(1+4a^2\rho^2)(\rho^2\rho'' - 2\rho(\rho')^2) + 4a^2\rho^3(\rho')^2 - \rho^3 = 0.$$

$$(1+4a^2\rho^2)(\rho\rho'' - 2(\rho')^2) + 4a^2\rho^2(\rho')^2 - \rho^2 = 0$$

$$2(\rho')^2(1+2a^2\rho^2) + \rho^2 - \rho\rho'' - 4a^2\rho^3\rho'' = 0.$$

#.

3. $$t = \int \frac{ds}{v(x,y,z)}$$

express $x = x(z)$, $y = y(z)$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$
$$= dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} = dz\sqrt{1 + (x')^2 + (y')^2}$$

$$\therefore f = \frac{\sqrt{1 + (x')^2 + (y')^2}}{v(x,y,z)} = \frac{g}{v(x,y,z)}$$ where $g = \sqrt{1 + (x')^2 + (y')^2}$

$$\frac{\partial f}{\partial x} = -\frac{1}{v^2}\frac{\partial v}{\partial x} g$$
$$\frac{\partial f}{\partial x'} = \frac{1}{v}\frac{\partial g}{\partial x'} = \frac{1}{2vg}\left[2x'\right] = \frac{x'}{vg}$$

similarly
$$\frac{\partial f}{\partial y} = -\frac{1}{v^2}\frac{\partial v}{\partial y} g$$
$$\frac{\partial f}{\partial y'} = \frac{1}{v}\frac{\partial g}{\partial y'} = \frac{y'}{vg}$$

now:
$$\frac{d}{dz}\left(\frac{\partial f}{\partial x'}\right) = \frac{(vg)x'' - x'\left(\frac{dv}{dz}g + v\frac{dg}{dz}\right)}{(vg)^2}$$

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[illegible]

$\frac{d}{dz} g \quad \frac{dg}{dz}$

$$= \left(\frac{\partial V}{\partial z} + \frac{\partial V}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial z}\right) g + \frac{V}{2g}\left[2x'x'' + 2y'y''\right].$$

$$= \left(\frac{\partial V}{\partial z} + \frac{\partial V}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial z}\right) g + \frac{V}{g}\left(x'x'' + y'y''\right)$$

$$\frac{d}{dz}\left(\frac{\partial f}{\partial x'}\right) = \frac{1}{V^2 g^3}\left\{ V g^2 x'' - x'\left(\frac{\partial V}{\partial z} + \frac{\partial V}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial z}\right) g^2 - x'V\left(x'x'' + y'y''\right)\right\}$$

$$= \frac{1}{V^2 g^3}\left[V g^2 x'' - x'\left(V_z + x'V_x + y'V_y\right) g^2 - x'V\left(x'x'' + y'y''\right)\right].$$

where $V_x = \frac{\partial V}{\partial x}$, etc.

$$\frac{d}{dz}\left(\frac{\partial f}{\partial y'}\right) = \frac{(Vg)\,y'' - y'\left(\frac{dV}{dz} g + V\frac{dg}{dz}\right)}{(Vg)^2}$$

$$= \frac{1}{V^2 g^3}\left[V g^2 y'' - y'\left(V_z + x'V_x + y'V_y\right) g^2 - y'V\left(x'x'' + y'y''\right)\right]$$

Euler - Lagrange Eqns

$$V g^2 x'' - x'\left(V_z + x'V_x + y'V_y\right) g^2 - x'V\left(x'x'' + y'y''\right) = -V_x\left[1 + (x')^2 + (y')^2\right]^2.$$

$$V g^2 y'' - y'\left(V_z + x'V_x + y'V_y\right) g^2 - y'V\left(x'x'' + y'y''\right) = -V_y\left[1 + (x')^2 + (y')^2\right]^2.$$

— a set of coupled ODEs.

4a. $\dot{y} = \frac{dy}{dt} = A \sin(kx)\, k \frac{dx}{dt}$

$= Ak(\cos kx)\, \dot{x}$

#

b. $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$

$= \frac{m}{2}\dot{x}^2\left[1 + (Ak\cos kx)^2\right] - mgA\sin kx.$

#.

c. $\frac{\partial L}{\partial \dot{x}} = m\dot{x}\left[1 + (Ak\cos kx)^2\right].$

$\frac{\partial L}{\partial x} = \frac{m\dot{x}^2}{2}\, 2(Ak\cos kx)\, Ak(-k\sin kx) - mgAk\cos kx.$

$= -mA^2k^3\dot{x}^2\cos kx \sin kx - mgAk\cos kx.$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}\left[1 + (Ak\cos kx)^2\right] + 2m\dot{x}(Ak\cos kx)(-Ak^2\sin kx\, \dot{x})$

$= m\ddot{x}\left[1 + (Ak\cos kx)^2\right] - 2mA^2k^3\dot{x}^2\cos kx \sin kx.$

Euler-Lagrange eqn.

$\ddot{x}\left[1 + (Ak\cos kx)^2\right] - A^2k^3\dot{x}^2\cos kx \sin kx + gAk\cos kx = 0.$

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1b. $f = \sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2} = \sqrt{1+(y')^2+(z')^2}$.

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{1}{2f}\,2y' = \frac{y'}{f}$$

$$\frac{\partial f}{\partial z} = 0 \quad , \quad \frac{\partial f}{\partial z'} = \frac{z'}{f}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{f y'' - y'\frac{df}{dx}}{f^2}$$

$$= \frac{f^2 y'' - y'\left[y'y'' + 4a^2(x+yy')(1+(y')^2+yy'')\right]}{f^3}$$

$$= \frac{y'' + \cancel{(y')^2 y''} + (z')^2 y'' - \cancel{(y')^2 y''} - 4a^2 y'(x+yy')(1+(y')^2+yy'')}{f^3}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = \frac{f z'' - z'\frac{df}{dx}}{f^2}$$

$$= \frac{f^2 z'' - z'\left[y'y'' + 4a^2(x+yy')(1+(y')^2+yy'')\right]}{f^3}$$

$$= z'' + (y')^2 z'' + (z')^2 z'' -$$

This is another method to solve 1b and 2b. This is much cleaner and the mess is only comes up in the final step so you don't have to carry it along with your derivation.
This method uses Lagrange's multiplier.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{f y'' - y' \frac{df}{dx}}{f^{2}}$$

$$\frac{df}{dx} = \frac{1}{2f}\, 2\left[y'y'' + z'z''\right] = \frac{y'y'' + z'z''}{f}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{f^2 y'' - y'(y'y'' + z'z'')}{f^3}$$

$$= \frac{y'' + \cancel{y''(y')^2} + y''(z')^2 - \cancel{(y')^2 y''} - y'z'z''}{f^3}$$

$$= \frac{y''(1+(z')^2) - y'z'z''}{f^3}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = \frac{f z'' - z' \frac{df}{dx}}{f^{2}}$$

$$= \frac{z'' + (y')^2 z'' + \cancel{(z')^2 z''} - z'y'y'' - \cancel{(z')^2 z''}}{f^3}$$

$$= \frac{z''(1+(y')^2) - z'y'y''}{f^3}$$

$$g(y, z, x) = z - a(x^2 + y^2).$$

$$\frac{\partial g}{\partial y} = -2ay \quad , \quad \frac{\partial g}{\partial z} = 1$$

$$\frac{y''(1+(z')^2) - y'z'z''}{f^3} + 2ay\lambda = 0$$

$$\frac{z''(1+(y')^2) - z'y'y''}{f^3} - \lambda = 0$$

eliminating $f^3\lambda$, and sub $z = a(x^2+y^2)$,

$$y''(1+(z')^2) - y'z'z'' + 2ay\left[z''(1+(y')^2) - z'y'y''\right] = 0$$

$$\cancel{y''\left[1+(2ax)^2\right]}$$

$$z' = \cancel{2ax+}\ 2a(x+yy')$$
$$z'' = 2a(1+(y')^2+yy'')$$

$$y''\left[1+4a^2(x+yy')^2\right] - y'4a^2(x+yy')(1+(y')^2+yy'')$$

$$+2ay\left[2a(1+(y')^2+yy'')(1+(y')^2) - 2a(x+yy')y'y''\right] = 0$$

✓

$$y''+4a^2y''(x^2+\cancel{2x}yy'+\cancel{y^2(y')^2}) - 4a^2(x+\cancel{yy'})y' - 4a^2(x+\cancel{yy'})(y')^3$$

$$-4a^2(\cancel{x}+\cancel{yy'})yy'y'' + 4a^2\left[y+y(y')^2+\cancel{y(y')^2}+\cancel{y(y')^4}+y^2y''+\cancel{y^2(y')^2y''}\right]$$

$$-4a^2y(\cancel{x}+\cancel{yy'})y'y'' = 0$$

$$\cancel{y''+4a}\quad y''(1+4a^2x^2+4a^2y^2) + 4a^2(y+y(y')^2 - xy' - x(y')^3) = 0$$

$$y''(1+4a^2x^2+4a^2y^2) + 4a^2(y-xy')(1+(y')^2) = 0.$$

5

2b. $f = \sqrt{\rho^2 + \left(\frac{d\rho}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2} = \sqrt{\rho^2 + (\rho')^2 + (z')^2}$

$f = f(\rho, z, \phi)$

$g(\rho, z, \phi) = z - a\rho^2$

$$\cancel{\frac{\partial f}{\partial \rho} = \frac{\rho + 4a^2\rho(\rho')^2}{f}}$$

$$\cancel{\frac{\partial f}{\partial \rho'} = \frac{\rho'(1 + 4a^2\rho^2)}{f}}$$

$$\frac{\partial f}{\partial \rho} = \frac{1}{2f}\left[2\rho\right] = \frac{\rho}{f}$$

$$\frac{\partial f}{\partial \rho'} = \frac{1}{2f}\left[2\rho'\right] = \frac{\rho'}{f}$$

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial \rho'}\right) = \frac{f\rho'' - \rho'\frac{df}{d\phi}}{f^2}$$

$$\frac{df}{d\phi} = \frac{1}{2f}\left[2\rho\rho' + 2\rho'\rho'' + 2z'z''\right] = \frac{\rho\rho' + \rho'\rho'' + z'z''}{f}$$

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial \rho'}\right) = \frac{f^2\rho'' - \rho'(\rho\rho' + \rho'\rho'' + z'z'')}{f^3}$$

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial \rho'}\right) - \frac{\partial f}{\partial \rho} - \lambda\frac{\partial g}{\partial \rho} = 0$$

$$\frac{\left[\rho^2 + (\rho')^2 + (z')^2\right]\rho'' - \rho'(\rho\rho' + \rho'\rho'' + z'z'')}{f^3} - \frac{\rho}{f} - \lambda(-2a\rho) = 0$$

$$\rho^2\rho'' + (z')^2\rho'' - \rho(\rho')^2 - \rho' z' z'' - \rho\left(\rho^2 + (\rho')^2 + (z')^2\right) + 2a\lambda\rho f^3 = 0$$

$$\rho^2\rho'' + (z')^2\rho'' - 2\rho(\rho')^2 - \rho' z' z'' - \rho^3 - \rho(z')^2 + 2a\lambda\rho f^3 = 0.$$

$$\frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial z'} = \frac{1}{2f}\left[2z'\right] = \frac{z'}{f}$$

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial z'}\right) = \frac{f z'' - z'\frac{df}{d\phi}}{f^2}$$

$$= \frac{f^2 z'' - z'(\rho\rho' + \rho'\rho'' + z'z'')}{f^3}$$

$$= \frac{\left[\rho^2 + (\rho')^2 + (z')^2\right]z'' - z'(\rho\rho' + \rho'\rho'' + z'z'')}{f^3}$$

$$= \frac{\rho^2 z'' + (\rho')^2 z'' - \rho\rho' z' - \rho'\rho'' z'}{f^3}$$

$$\frac{\rho^2 z'' + (\rho')^2 z'' - \rho\rho' z' - \rho'\rho'' z'}{f^3} - \lambda(1) = 0.$$

$$\rho^2 z'' + (\rho')^2 z'' - \rho\rho' z' - \rho'\rho'' z' - \lambda f^3 = 0.$$

eliminating λf_0^3,

$\cancel{\rho^2 z'' + (\rho')^2 z}$

$$\rho^2\rho'' + (z')^2\rho'' - 2\rho(\rho')^2 - \rho' z' z'' - \rho^3 - \rho(z')^2$$

$$+ 2a\rho\left[\rho^2 z'' + (\rho')^2 z'' - \rho\rho' z' - \rho'\rho'' z'\right] = 0$$

use a fact $\quad g(\rho, z, \rho') = 0$

$$z = a\rho^2$$

$$z' = 2a\rho\rho' , \quad z'' = 2a\left[(\rho')^2 + \rho\rho''\right]$$

$$\rho^2\rho'' + 4a^2\rho^2(\rho')^2\rho'' - 2\rho(\rho')^2 - \rho'(2a\rho\rho')(2a)((\rho')^2 + \rho\rho'') - \rho^3$$

$$-\rho\, 4a^2\rho^2(\rho')^2 + 2a\rho\Big[2a\rho^2\left[(\rho')^2 + \rho\rho''\right] + (\rho')^2 2a\left[(\rho')^2 + \rho\rho''\right]$$

$$- \rho\rho'(2a\rho\rho') - \rho'\rho''(2a\rho\rho')\Big] = 0$$

$$-\rho^3 + \rho^2\rho'' + 4a^2\cancel{\rho^2(\rho')^2\rho''} - 2\rho(\rho')^2 - 4a^2\cancel{\rho(\rho')^4} - 4a^2\cancel{\rho^2(\rho')^2\rho''}$$

$$-4a^2\cancel{\rho^3(\rho')^2} + 4a^2\cancel{\rho^3(\rho')^2} + 4a^2\rho^4\rho'' + 4a^2\cancel{\rho(\rho')^4} + 4a^2\cancel{\rho^2(\rho')^2}\rho''$$

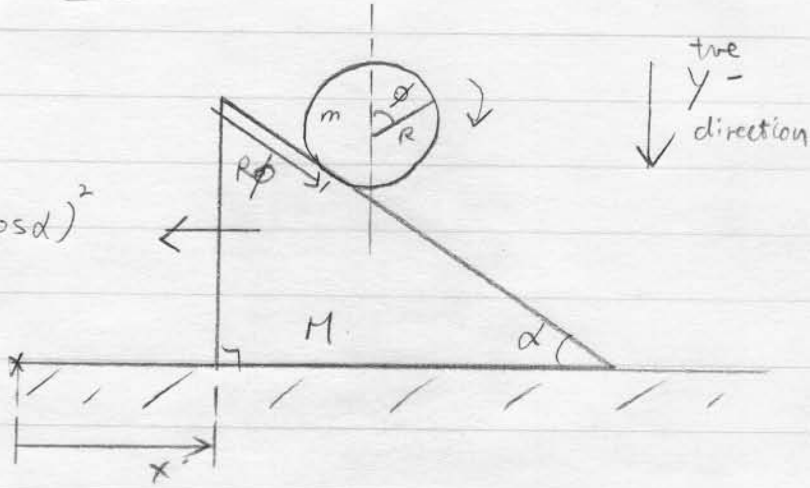
$$-4a^2\rho^3(\rho')^2 - 4a^2\rho^2\cancel{(\rho')^2}\rho'' = 0$$

$$2(\rho')^2(1 + 2a^2\rho^2) + \rho^2 - \rho\rho'' - 4a^2\rho^3\rho'' = 0$$

PHYS 410

Hw2 <u>solns</u>

7-6.



KE $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + R\dot{\phi}\cos\alpha)^2 + (R\dot{\phi}\sin\alpha)^2\right] + \frac{1}{2}I\dot{\phi}^2$.

for hoop, moment of inertia $I = mR^2$.

$= -mgR\phi\sin\alpha$. where top of hill is set as zero PE

$$L = T - V$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\phi}^2 + mgR\phi\sin\alpha + mR\dot{x}\dot{\phi}\cos\alpha.$$

$$= \frac{1}{2}M\dot{x}^2 + mR^2\dot{\phi}^2 + mgR\phi\sin\alpha + mR\dot{x}\dot{\phi}\cos\alpha.$$

since L indpt. of x, t.

$$p_x = \frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + mR\dot{\phi}\cos\alpha = \text{const}. \#$$

energy $E = \frac{1}{2}M\dot{x}^2 + mR^2\dot{\phi}^2 - mgR\phi\sin\alpha + mR\dot{x}\dot{\phi}\cos\alpha = \text{const}.$

\#

Lagrange's eqns.

$$(M+m)\ddot{x} + mR\ddot{\phi}\cos\alpha = 0.$$

\#.

$$\frac{\partial L}{\partial \dot{\phi}} = 2mR^2\dot{\phi} + mR\dot{x}\cos\alpha$$

$$\frac{\partial L}{\partial \phi} = mgR\sin\alpha .$$

$$2mR^2\ddot{\phi} + mR\ddot{x}\cos\alpha = mgR\sin\alpha$$

$$2R\ddot{\phi} + \ddot{x}\cos\alpha = g\sin\alpha$$

$$\frac{\partial L}{\partial \dot\phi} = 2mR^2\dot\phi + mR\dot x\cos\alpha$$

$$\frac{\partial L}{\partial \phi} = mgR\sin\alpha.$$

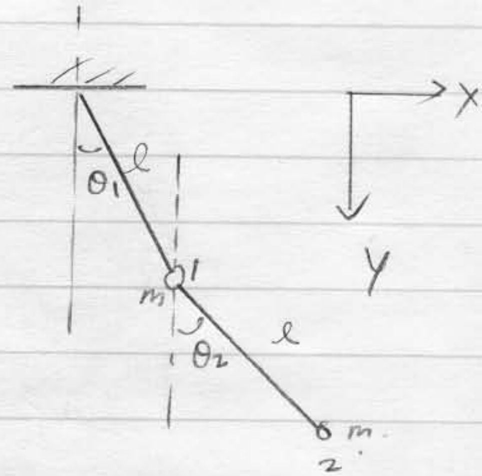
$$2mR^2\ddot\phi + mR\ddot x\cos\alpha = mgR\sin\alpha$$

$$2R\ddot\phi + \ddot x\cos\alpha = g\sin\alpha$$

7-7.

$$x_1 = l\sin\theta_1$$
$$y_1 = l\cos\theta_1$$

$$x_2 = l\sin\theta_1 + l\sin\theta_2$$
$$y_2 = l\cos\theta_1 + l\cos\theta_2.$$



KE $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$

$$= \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

PE $V = -mgy_1 - mgy_2.$

$$L = T - V$$
$$= \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$
$$+ mgl(2\cos\theta_1 + \cos\theta_2).$$

#.

Lagrange's eqns given by.

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} m l^2 \left[4\dot{\theta}_1 + 2\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right].$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} m l^2 \left[2\dot{\theta}_2 + 2\dot{\theta}_1 \cos(\theta_1 - \theta_2) \right]$$

$$\frac{\partial L}{\partial \theta_1} = -m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - 2mgl \sin\theta_1 .$$

$$\frac{\partial L}{\partial \theta_2} = m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - mgl \sin\theta_2 .$$

$$\frac{1}{2} m l^2 \left[4\ddot{\theta}_1 + 2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - 2\dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \right]$$

$$= -m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - 2mgl \sin\theta_1 .$$

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \frac{2g}{l} \sin\theta_1 = 0 .$$

#

$$\frac{1}{2} m l^2 \left[2\ddot{\theta}_2 + 2\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - 2\dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \right]$$

$$= m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - mgl \sin\theta_2 .$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{g}{l} \sin\theta_2 = 0 .$$

#

a. This is straightforward and just follow similar lines I did in part (b). So I am going to show you the result in (a) by taking the limit $m \to 0$ in (b).

Since $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

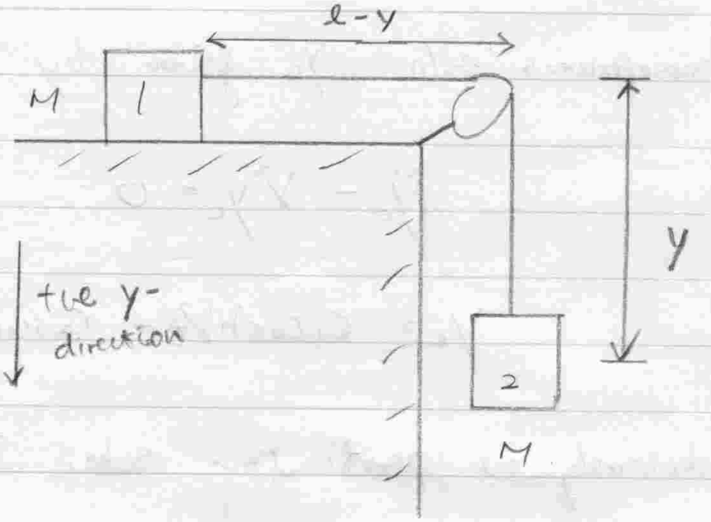
$$y = \frac{Ml}{m}\left[\left[1 + \frac{(\gamma t)^2}{2!} + \frac{(\gamma t)^4}{4!} + \cdots\right] - 1\right]$$

recall $\gamma^2 \propto \dfrac{m}{2M+m}$, so in the limit $m \to 0$, terms with $O(\gamma^4)$ vanishes, so

$$y = \frac{Ml}{2}\,\frac{g}{2Ml}\,t^2 = \frac{g}{4}t^2$$

#.

7-10b. velocity of mass 1 $= \dot{\overline{l-y}} = -\dot{y}$



KE $T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}M\dot{y}^2$

$+ \frac{1}{2}m\dot{y}^2$

$= (M + \frac{m}{2})\dot{y}^2$

PE $V = -Mgy - (m\frac{y}{l})g\frac{y}{2}$. since CM of hanging rope is half way down the length of rope.

$L = (M+\frac{m}{2})\dot{y}^2 + Mgy + m\frac{g}{2l}y^2$.

$\frac{\partial L}{\partial \dot{y}} = 2M\dot{y} + m\dot{y}$

$\frac{\partial L}{\partial y} = Mg + m\frac{g}{l}y$.

Lagrange's eqn.

$2M\ddot{y} + m\ddot{y} = Mg + m\frac{g}{l}y$.

$\ddot{y} - \frac{mg}{l(2M+m)}y - \frac{g}{2M+m} = 0$

$\ddot{y} - \gamma^2 y - \frac{M}{m}\gamma^2 l = 0$ where $\gamma = \sqrt{\frac{mg}{l(2M+m)}}$

homogeneous soln. y_c given by

$$\ddot{y}_c - \gamma^2 y_c = 0.$$

$$y_c = C_1 \cosh \gamma t + C_2 \sinh \gamma t.$$

obviously one particular soln. is given by

$$-\gamma^2 y_p - \frac{M}{m}\gamma^2 l = 0$$

$$y_p = -\frac{M}{m} l.$$

$\therefore$ general soln.

$$y = y_c + y_p$$

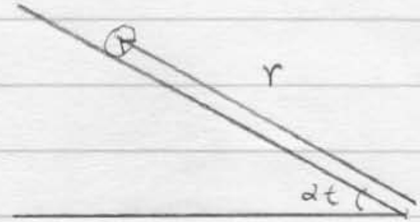
$$= C_1 \cosh \gamma t + C_2 \sinh \gamma t - \frac{M}{m} l.$$

assume $y(0) = 0$, $y'(0) = 0$

$\Rightarrow$ $C_1 = \frac{M}{m} l$, $C_2 = 0$.

$\therefore$ $y(t) = \frac{Ml}{m}(\cosh \gamma t - 1)$. #

7-12.



KE $T = \frac{1}{2}m(\dot{r}^2 + r^2\alpha^2)$.

PE = $V = mgr\sin\alpha t$.

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial r} = mr\alpha^2 - mg\sin\alpha t.$$

Lagrange's eqn.

$$m\ddot{r} = mr\alpha^2 - mg\sin\alpha t.$$

$$\ddot{r} - \alpha^2 r + g\sin\alpha t = 0.$$

(Optional.)

---

homogeneous soln. given by $\ddot{r}_c - \alpha^2 r_c = 0$.

$$r_c = C_1\cosh\alpha t + C_2\sinh\alpha t$$

particular soln. given by

$$r_p = \frac{1}{D^2 - \alpha^2}(-g\sin\alpha t) \quad \text{where } D \equiv \frac{d}{dt}$$

$$= \frac{1}{(D+\alpha)(D-\alpha)}(-g\sin\alpha t) \quad \text{is the differential operator.}$$

Consider $\frac{1}{D^2 - \alpha^2} e^{i\alpha t}$.

general sol'n.

$$\begin{aligned} r &= r_c + r_p \\ &= C_1 \cosh \alpha t + C_2 \sinh \alpha t + \frac{g}{2\alpha^2} \sin \alpha t . \end{aligned}$$

$$r\big|_{t=0} \equiv r_0 = C_1 .$$

$$\dot{r}\big|_{t=0} = \alpha \left(C_1 \sinh \alpha t + C_2 \cosh \alpha t \right) + \frac{g}{2\alpha} \cos \alpha t \Big|_{t=0} = 0$$

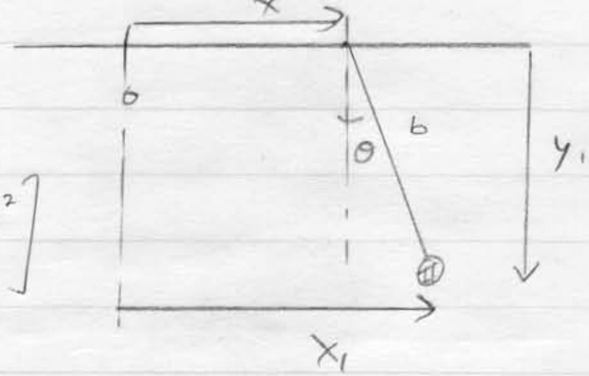
$$C_2 = -\frac{g}{2\alpha^2} .$$

$$\therefore \quad r = r_0 \cosh \alpha t + \frac{g}{2\alpha^2} \left(\sin \alpha t - \sinh \alpha t \right)$$

#

7-16. $x_1 = x + b\sin\theta$.

$y_1 = b\cos\theta$.



KE $T = \frac{1}{2} m \left[(\dot{x} + b\dot{\theta}\cos\theta)^2 + (b\dot{\theta}\sin\theta)^2 \right]$

$= \frac{1}{2} m \left[\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta \right]$

PE $V = -mgb\cos\theta$.

$L = \frac{1}{2} m \left[\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta \right] + mgb\cos\theta$

$= \frac{1}{2} m \left[b^2\dot{\theta}^2 + \omega^2 a^2 \cos^2\omega t + 2ab\omega\dot{\theta}\cos\theta\cos\omega t \right] + mgb\cos\theta$.

$\frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} + mab\omega\cos\omega t\cos\theta$

$\frac{\partial L}{\partial \theta} = -mab\omega\dot{\theta}\cos\omega t\sin\theta - mgb\sin\theta$.

Lagrange's eqn.

$b^2\ddot{\theta} - ab\omega^2\sin\omega t\cos\theta - ab\omega\dot{\theta}\cos\omega t\sin\theta$

$= -ab\omega\dot{\theta}\cos\omega t\sin\theta - gb\sin\theta$

$\ddot{\theta} - \frac{a}{b}\omega^2\sin\omega t\cos\theta + \frac{g}{b}\sin\theta = 0$

#.

HW3 Soln.

7.23. $H = \frac{|\vec{p}|^2}{2m} + U$

$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + U$

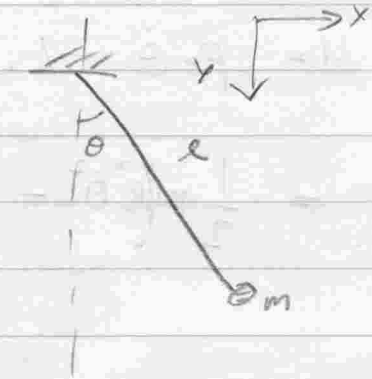
$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}.$

$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}, \quad \dot{p}_y = -\frac{\partial U}{\partial y}, \quad \dot{p}_z = -\frac{\partial U}{\partial z}.$

$\Rightarrow m\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla} U$

#.

7-24. Choosing origin at the suspension point,



$$x = \ell \sin\theta$$
$$y = +\ell\cos\theta$$

$$\dot x = \dot\ell \sin\theta + \dot\theta \ell \cos\theta$$
$$\dot y = +\dot\ell \cos\theta - \dot\theta \ell \sin\theta$$

$$T = \frac{1}{2} m (\dot x^2 + \dot y^2)$$
$$= \frac{1}{2} m \left[\dot\ell^2 + \dot\theta^2 \ell^2 \right]$$

$$U = -mgy$$
$$= -mg\ell\cos\theta$$

for const. $\dot\ell = -\alpha$, $\ell(t) = -\alpha t + \beta$, ($\beta$ const)

$$L = \frac{1}{2} m \left[\dot\ell^2 + \dot\theta^2 \ell^2 \right] + mg\ell\cos\theta$$
$$= \frac{1}{2} m \left(\ell^2 \dot\theta^2 + \alpha^2 \right) + mg\ell\cos\theta = \frac{1}{2} m \left[(-\alpha t + \beta)^2 \dot\theta^2 + \alpha^2 \right] + mg(-\alpha t + \beta)\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m\ell^2 \dot\theta$$ angular momentum of ball

$H = p_\theta \dot{\theta} - L$

$$= \frac{1}{2}m(l^2\dot{\theta}^2 - \alpha^2) - mgl\cos\theta = \frac{1}{2}m\left[(-\alpha t+\beta)^2\dot{\theta}^2 - \alpha^2\right] - mg(-\alpha t+\beta)\cos\theta$$

#

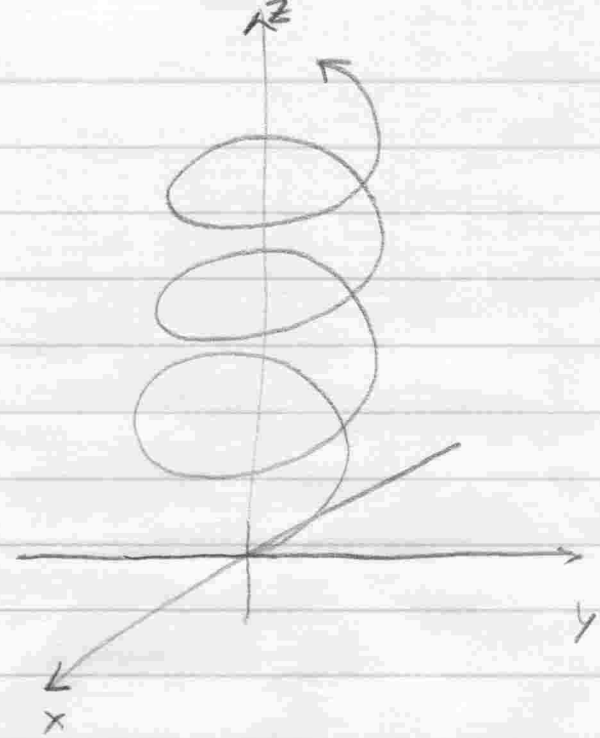
However,

energy $E = \frac{1}{2}m(l^2\dot{\theta}^2 + \alpha^2) - mgl\cos\theta \neq H$

#

Since L and H are explicitly time-dept, energy is not conserved and hence the Hamiltonian does not represent the energy of the system.

7-25 Cylindrical coord.



$r = \text{const}$

$z = k\theta$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2).$$

$\dot{r} = 0, \quad \dot{z} = k\dot{\theta}$

$$\Rightarrow T = \frac{1}{2}m\left(r^2\dot{\theta}^2 + k^2\dot{\theta}^2\right)$$

$$= \frac{m}{2}(r^2+k^2)\dot{\theta}^2.$$

$$U = mgz = mgk\theta.$$

$$L = \frac{1}{2}m(r^2+k^2)\dot{\theta}^2 - mgk\theta.$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(r^2+k^2)\dot{\theta}$$

$$H = \dot{\theta}p_\theta - L$$

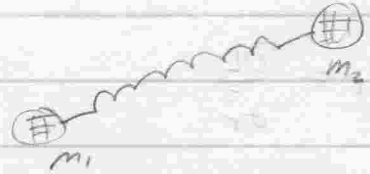
$$= \frac{1}{2}m(r^2+k^2)\dot{\theta}^2 + mgk\theta = \frac{p_\theta^2}{2m(r^2+k^2)} + mgk\theta.$$

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{m(r^2+k^2)} \\[2ex] \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = -mgk. \end{cases}$$

$\left(\begin{array}{l}\text{thus } \theta \propto t^2 \\ \quad\quad z \propto t^2\end{array}\right)$

#

7-27. since system doesn't translate as a whole, we can choose the origin as the centre of mass.



then $T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

$U = \frac{1}{2}k(r-b)^2$

$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-b)^2$.

$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$, $\frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$

$\frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - k(r-b)$, $\frac{\partial L}{\partial \theta} = 0$.

Lagrange's eqns:

$\mu\ddot{r} = \mu r\dot{\theta}^2 - k(r-b)$

$\mu(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = 0 \Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$.

#.

b. θ is the cyclic coordinate.

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$

#.

c. $p_r = \frac{\partial L}{\partial \dot{r}}$

$= \mu \dot{r}$

$H = \dot{r} p_r + \dot{\theta} p_\theta - L$

$= \frac{1}{2}(\mu \dot{r}^2 + \mu r^2 \dot{\theta}^2) + \frac{1}{2} k (r-b)^2 = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + \frac{1}{2} k (r-b)^2.$

$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}.$

$\dot{p}_r = -\frac{\partial H}{\partial r} = -k(r-b) + \frac{p_\theta^2}{\mu r^3}.$

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}.$

$\dot{p}_\theta = 0$

#.

7-30a. $[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$

$$\frac{dg}{dt} = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial g}{\partial p_k} \frac{dp_k}{dt} \right) + \frac{\partial g}{\partial t}.$$

since g has dependence on time through q_k, p_k as well as an explicit dependence on time.

by Hamilton's eqns, $\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}$, $\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}$.

$$= \sum_k \left[\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right] + \frac{\partial g}{\partial t}$$

$$= [g, H] + \frac{\partial g}{\partial t}$$

#.

b. follows straightforwardly from (a) if we put $g = q_j$ and $g = p_j$ in turn.

c. $[p_i, p_j] = \cancel{\frac{\partial p_i}{\partial q_j}}^{\,0} \frac{\partial p_j}{\partial p_j} - \frac{\partial p_i}{\partial p_i} \cancel{\frac{\partial p_j}{\partial q_i}}^{\,0} = 0.$

$[q_i, q_j] = \frac{\partial q_i}{\partial q_i} \cancel{\frac{\partial q_j}{\partial p_i}}^{\,0} - \cancel{\frac{\partial q_i}{\partial p_j}}^{\,0} \frac{\partial q_j}{\partial q_j} = 0$

d. $[q_i\, p_j]$

$$= \sum_k \left(\underbrace{\frac{\partial q_i}{\partial q_k}}_{\delta_{ik}} \underbrace{\frac{\partial p_j}{\partial p_k}}_{\delta_{jk}} - \cancelto{0}{\frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k}} \right)$$

$$= \delta_{ij} \quad \#.$$

e. quantity f has no explicit time dept $\Rightarrow$ $\frac{\partial f}{\partial t} = 0$

commute with H $\Rightarrow$ $[f, H] = 0$

from (a), $\frac{df}{dt} = 0$ hence f is a const. of motion.

#.

HW 4 soln.

1.a. The effective potential is

$$V_{eff} = \frac{-k}{r^\gamma} + \frac{l^2}{2\mu r^2}$$

hence the eqn. of motion for r is

$$\mu \ddot{r} = -\frac{d}{dr}\left(-\frac{k}{r^\gamma} + \frac{l^2}{2\mu r^2}\right)$$

for circular orbits, $\ddot{r} = \dot{r} = 0$, so

$$\frac{d}{dr}\left(-\frac{k}{r^\gamma} + \frac{l^2}{2\mu r^2}\right) = 0$$

$$\frac{\gamma k}{r^{\gamma+1}} - \frac{l^2}{\mu r^3} = 0$$

$$r = \left(\frac{l^2}{\mu \gamma k}\right)^{\frac{1}{2-\gamma}}$$

#.

b. the eqn. of motion for θ is

$$\dot{\theta}(t) = \frac{l}{\mu r^2}.$$

for circular motion, $\dot{\theta} = \omega = \frac{2\pi}{T}$

$$T = \frac{2\pi \mu r^2}{l} = \frac{2\pi\mu}{l}\left(\frac{l^2}{\mu \gamma k}\right)^{\frac{2}{2-\gamma}}$$

#.

c. by applying a small perturbation δr, the otherwise circular orbit of the planet executes a small oscillation in the radial direction,

put, $r = r_o + \delta r$ where r_o = radius of circular orbit

$$\mu(\ddot{r}_o + \delta\ddot{r}) = -\frac{d}{dr}\left(\frac{-k}{(r_o+\delta r)^{\gamma}} + \frac{l^2}{2\mu(r_o+\delta r)^2}\right)$$

$$\mu\delta\ddot{r} = -\frac{\gamma k}{(r_o+\delta r)^{\gamma+1}} + \frac{l^2}{\mu(r_o+\delta r)^3}.$$

expand $\left(1+\frac{\delta r}{r_o}\right)^{-(\gamma+1)}$ and $\left(1+\frac{\delta r}{r_o}\right)^{-3}$ up to $O\left(\frac{\delta r}{r_o}\right)$

$$\mu\delta\ddot{r} = -\frac{\gamma k}{r_o^{\gamma+1}}\left[1-(\gamma+1)\frac{\delta r}{r_o}\right] + \frac{l^2}{\mu r_o^3}\left[1-3\frac{\delta r}{r_o}\right].$$

$$\mu\delta\ddot{r} = \left[\frac{\gamma(\gamma+1)k}{r_o^{\gamma+2}} - \frac{3\gamma k}{r_o^{\gamma+2}}\right]\delta r \quad \#$$

from which we have used the condition for circular orbit # in (a).

$$\delta\ddot{r} = \frac{(\gamma^2+\gamma-3\gamma)k}{\mu r_o^{\gamma+2}}\delta r$$

$$= \frac{\gamma(\gamma-2)k}{\mu r_o^{\gamma+2}}\delta r.$$

#.

c. by applying a small perturbation δr, the otherwise circular orbit of the planet executes a small oscillation in the radial direction,

put, $r = r_0 + \delta r$ where r_0 = radius of circular orbit

$$\mu(\ddot{r}_0 + \delta\ddot{r}) = -\frac{d}{dr}\left(\frac{-k}{(r_0+\delta r)^{\gamma}} + \frac{l^2}{2\mu(r_0+\delta r)^2}\right)$$

$$\mu\delta\ddot{r} = -\frac{\gamma k}{(r_0+\delta r)^{\gamma+1}} + \frac{l^2}{\mu(r_0+\delta r)^3}.$$

expand $\left(1+\frac{\delta r}{r_0}\right)^{-(\gamma+1)}$ and $\left(1+\frac{\delta r}{r_0}\right)^{-3}$ up to $O\left(\frac{\delta r}{r_0}\right)$

$$\mu\delta\ddot{r} = -\frac{\gamma k}{r_0^{\gamma+1}}\left[1-(\gamma+1)\frac{\delta r}{r_0}\right] + \frac{l^2}{\mu r_0^3}\left[1-3\frac{\delta r}{r_0}\right].$$

$$\mu\delta\ddot{r} = \left[\frac{\gamma(\gamma+1)k}{r_0^{\gamma+2}} - \frac{3\gamma k}{r_0^{\gamma+2}}\right]\delta r$$ from which we have used the condition for circular orbit in (a).

$$\delta\ddot{r} = \frac{(\gamma^2+\gamma-3\gamma)k}{\mu r_0^{\gamma+2}}\delta r$$

$$= \frac{\gamma(\gamma-2)k}{\mu r_0^{\gamma+2}}\delta r.$$

#.

d. The period is obviously given by (for radial oscillation)

$$\omega^2 = \frac{\gamma(2-\gamma)k}{\mu r_0^{\gamma+2}}.$$

$$\frac{2\pi}{T_r} = \left[\frac{\gamma(2-\gamma)k}{\mu r_0^{\gamma+2}}\right]^{\frac{1}{2}}.$$

$$T_r = 2\pi\left[\frac{\mu r_0^{\gamma+2}}{\gamma(2-\gamma)k}\right]^{\frac{1}{2}}$$

#.

now the angle swept by the radius vector in one radial oscillation is

$$\Delta\theta = \omega T_r$$

$$= \left[\frac{l}{\mu r_0^2} 2\pi\right]\left[\frac{\mu r_0^{\gamma+2}}{\gamma(2-\gamma)k}\right]^{\frac{1}{2}}$$

where from (b) $\omega = \frac{l}{\mu r_0^2}$.

$$= 2\pi\left(\frac{l^2}{\mu k}\right)^{\frac{1}{2}}\left(\frac{1}{r_0^{2-\gamma}}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\gamma(2-\gamma)}}$$

$$= 2\pi\left(\gamma r_0^{2-\gamma}\right)^{\frac{1}{2}}\left(\frac{1}{r_0^{2-\gamma}}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\gamma(2-\gamma)}}$$

where the condition for circular orbit is used.

$$= \frac{2\pi}{\sqrt{2-\gamma}}$$

for closed orbit, require $\frac{1}{\sqrt{2-\gamma}}$ = rational which is in no way satisfied other than $\gamma = 1$ #.

$$\mu \delta \ddot{r} = -k\gamma r_0^{\gamma-1}\left[1+(\gamma-1)\frac{\delta r}{r_0}\right] + \frac{l^2}{\mu r_0^3}\left[1-\frac{3\delta r}{r_0}\right]$$

$$= \left[-k\gamma r_0^{\gamma-2}(\gamma-1) - \frac{3l^2}{\mu r_0^4}\right]\delta r.$$

$$= \left[-k\gamma r_0^{\gamma-2}(\gamma-1) - 3k\gamma r_0^{\gamma-2}\right]\delta r.$$

$$= -\gamma(\gamma+2)k r_0^{\gamma-2}\delta r.$$

$$\delta\ddot{r} = -\frac{\gamma(\gamma+2)k r_0^{\gamma-2}}{\mu}\delta r.$$

#.

d. $$\omega^2 = \frac{\gamma(\gamma+2)k r_0^{\gamma-2}}{\mu}$$

$$T_r = 2\pi\left[\frac{\mu}{\gamma(\gamma+2)k r_0^{\gamma-2}}\right]^{\frac{1}{2}}$$

#.

$$\Delta\theta = \omega T_r$$

$$= \left(\frac{l}{\mu r_0^2}\right)2\pi\left[\frac{\mu}{\gamma(\gamma+2)k r_0^{\gamma-2}}\right]^{\frac{1}{2}}$$

$$= 2\pi\left(\frac{l^2}{\mu k}\right)^{\frac{1}{2}}\left(\frac{1}{r_0^{\gamma+2}}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\gamma(\gamma+2)}}$$

$$= 2\pi\,\gamma^{\frac{1}{2}} r_0^{\frac{\gamma+2}{2}}\frac{1}{r_0^{\frac{\gamma+2}{2}}}\frac{1}{\sqrt{\gamma(\gamma+2)}}$$

$$= \frac{2\pi}{\sqrt{\gamma+2}}$$

for closed orbit, then $\gamma = 2$ for $\frac{1}{\sqrt{\gamma+2}}$ to be rational.

8.2 $$\theta(r) = \int \frac{\frac{\ell}{r^2}\, dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}}$$

where the origin of θ has been defined in such a way to eliminate the integration constant, when min. of r is at $\theta = 0$.

put $u = \frac{1}{r}$.

$$\theta(u) = -\int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2} u - u^2}}$$

from table, $$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left[-\frac{\beta + 2\gamma x}{\sqrt{q}}\right]$$

where $q = \beta^2 - 4\alpha\gamma$.

so setting $\alpha = \frac{2\mu E}{\ell^2}$, $\beta = \frac{2\mu k}{\ell^2}$, $\gamma = -1$.

$$q = \left(\frac{2\mu k}{\ell^2}\right)^2 \left(1 + \frac{2E\ell^2}{\mu k^2}\right)$$

and

$$\theta(u) = -\cos^{-1}\left[\frac{\frac{\ell^2 u}{\mu k} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}\right]$$

$$\cos\theta(r) = \frac{\frac{l^2}{\mu k}\frac{1}{r} - 1}{\sqrt{1+\frac{2El^2}{\mu k^2}}}$$

#.

8.8. $$\theta(r) = \int \frac{\frac{l}{r^2}\,dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$$

now for $F(r) = kr$, $U(r) = -\frac{1}{2}kr^2$

thus

$$\theta(r) = \int \frac{\frac{l}{r^2}\,dr}{\sqrt{2\mu\left(E + \frac{1}{2}kr^2 - \frac{l^2}{2\mu r^2}\right)}}$$

$$= \int \frac{l\,dr}{r\sqrt{2\mu\left(Er^2 + \frac{1}{2}kr^4 - \frac{l^2}{2\mu}\right)}}$$

so it is appealing to sub. $u = r^2$; $du = 2r\,dr$

$$\theta(u) = \int \frac{l\,\frac{du}{2r}}{r\sqrt{2\mu\left[Eu + \frac{1}{2}ku^2 - \frac{l^2}{2\mu}\right]}}$$

$$= \frac{l}{2}\int \frac{du}{u\sqrt{2\mu\left[Eu + \frac{1}{2}ku^2 - \frac{l^2}{2\mu}\right]}}$$

from which it is appealing to sub. $w = \frac{1}{u}$ to further simplify

$$\theta(w) = \frac{\ell}{2}\int \frac{-u^2 dw}{u\sqrt{2\mu\left[E\frac{1}{w} + \frac{1}{2}k\frac{1}{w^2} - \frac{\ell^2}{2\mu}\right]}}$$

$$= \frac{-\ell}{2\sqrt{2\mu}}\int \frac{dw}{\sqrt{\frac{k}{2} + Ew - \frac{\ell^2}{2\mu}w^2}}$$

apply the formula in $\S$-2 with the sub.

$$\alpha = \frac{k}{2}, \quad \beta = E, \quad \gamma = -\frac{\ell^2}{2\mu}$$

$$q = E^2 + 4\frac{k}{2}\frac{\ell^2}{2\mu} = E^2 + \frac{k\ell^2}{\mu}$$

then

$$\theta(w) = \frac{-1}{\sqrt{\frac{\ell^2}{2\mu}}}\frac{\ell}{2\sqrt{2\mu}}\cos^{-1}\left[-\frac{E - \frac{\ell^2}{\mu}w}{\sqrt{E^2 + \frac{k\ell^2}{\mu}}}\right]$$

$$\cos[2\theta(r)] = -\frac{E - \frac{\ell^2}{\mu r^2}}{\sqrt{E^2 + \frac{k\ell^2}{\mu}}} = \frac{\frac{\ell^2}{\mu E r^2} - 1}{\sqrt{1 + \frac{k\ell^2}{\mu E^2}}}$$

similar to the Coulomb potential case we can define

$$\alpha \equiv \frac{\ell^2}{\mu E}, \quad \epsilon = \sqrt{1 + \frac{k\ell^2}{\mu E^2}}$$

then

$$\cos 2\theta = \frac{\frac{\alpha}{r^2} - 1}{\epsilon}.$$

$$\epsilon(\cos^2\theta - \sin^2\theta) = \frac{\alpha}{r^2} - 1$$

$$\epsilon r^2(\cos^2\theta - \sin^2\theta) + r^2 = \alpha$$

now polar $\rightarrow$ Cartesian : $x = r\cos\theta$, $y = r\sin\theta$

$$\epsilon(x^2 - y^2) + x^2 + y^2 = \alpha.$$

$$(\epsilon + 1)x^2 + (1 - \epsilon)y^2 = \alpha.$$

$$\frac{x^2}{\left(\frac{1}{\epsilon+1}\right)} - \frac{y^2}{\left(\frac{1}{\epsilon-1}\right)} = \alpha. \qquad \text{since } \epsilon = \sqrt{1 + \cdots} > 1$$

— hyperbolic orbit.

#.

8-10. At the instant ~~the~~ half of the mass of the sun was taken away, the velocity of the earth would remain unchanged. Since $M_{sun} \gg m_E$, we are assuming the Sun is stationary and the Earth moves in a circular orbit around it before the mass - disappearance occurred

Before : $E_0 = T + U$

$$= \frac{1}{2} m_E V_E^2 - \frac{G M_{sun} m_E}{R}$$

since $$\frac{m_E V_E^2}{R} = \frac{G M_{sun} m_E}{R^2}$$

$$E_0 = - \frac{G M_{sun} m_E}{2R}.$$

After : $E = T + U$

$$= \frac{1}{2} \frac{G M_{sun} m_E}{R} - \frac{G \left(\frac{M_{sun}}{2}\right) m_E}{R}$$

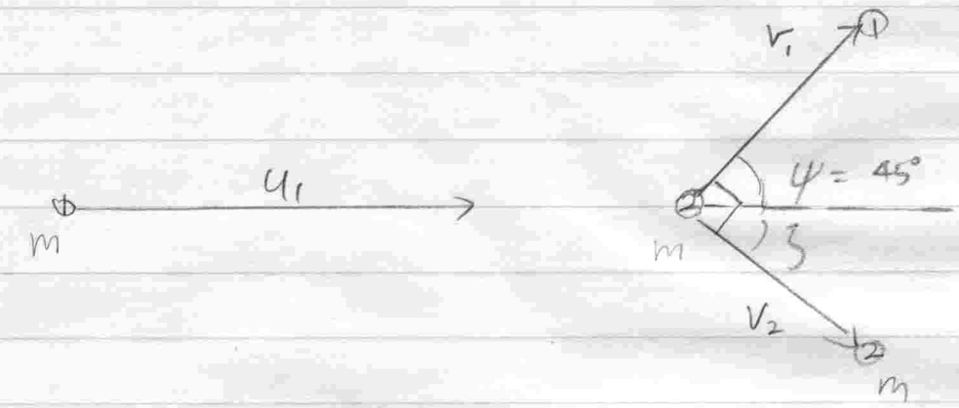
$$= 0.$$

recalling
$$\epsilon = \sqrt{1 + \frac{2 E l^2}{\mu k^2}} \quad \text{for } \frac{1}{r} \text{ potential}$$

we see that $\epsilon = 1$ and hence the Earth will fly off in a parabolic orbit.

HW·5 Soln.

8-34



For elastic collisions btn two same masses, the two emerge with an angle 90° btn them, so $\zeta = 45°$.

it follows that by conservation of linear momentum in the y-dir,

$$m v_1 \sin 45° = m v_2 \sin 45° \Rightarrow v_1 = v_2$$

or you can easily see this from symmetry of the configuration.

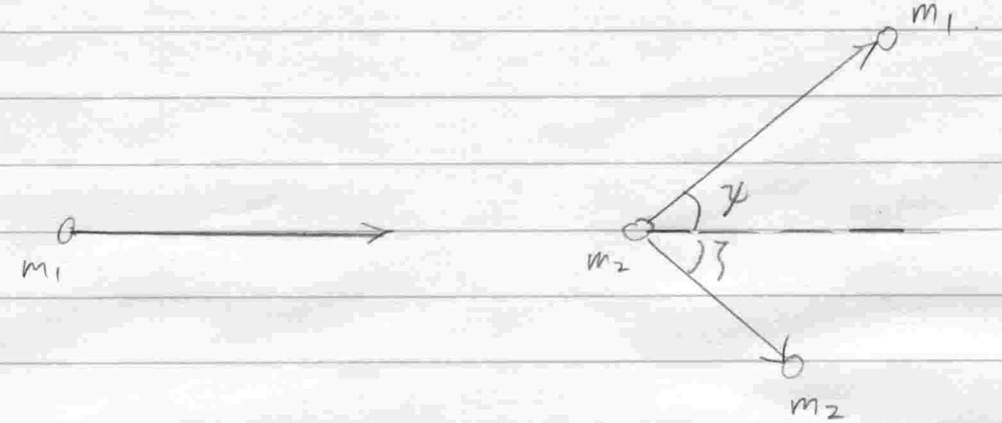
finally we have conservation of energy

$$\frac{1}{2} m u_1^2 = 2 \cdot \frac{1}{2} m v_1^2 \Rightarrow v_1 = \frac{u_1}{\sqrt{2}}.$$

$$v_2 = \frac{u_1}{\sqrt{2}}$$

∎.

P. 27.



For elastic collisions, K.E. and linear momentum are conserved. In particular, since the total momentum before collision is zero, it should be zero also afterwards,

$$p_1 \sin\psi = p_2 \sin\zeta$$

now $T_1 = \dfrac{p_1^2}{2m_1} \Rightarrow p_1 = \sqrt{2m_1 T_1}$

similarly $p_2 = \sqrt{2m_2 T_2}$.

note that T_1, T_2 are conserved also. Therefore

$$\frac{\sin\zeta}{\sin\psi} = \frac{p_2}{p_1} = \sqrt{\frac{m_2 T_2}{m_1 T_1}}$$

答

P-45 $$\sigma(\psi) = \sigma(\theta)\cdot\frac{[x\cos\psi + \sqrt{1-x^2\sin^2\psi}]}{\sqrt{1-x^2\sin^2\psi}}$$

$$\theta = \sin^{-1}(x\sin\psi) + \psi$$

$$\Rightarrow x = \frac{\sin(\theta-\psi)}{\sin\psi}$$

~~RHS~~ consider $$\frac{\sigma(\psi)}{\sigma(\theta)} = \frac{[x\cos\psi + \sqrt{1-x^2\sin^2\psi}]^2}{\sqrt{1-x^2\sin^2\psi}}$$

$$1 - x^2\sin^2\psi = 1-\sin^2(\theta-\psi) = \cos^2(\theta-\psi)$$

$$RHS = \frac{\left[\frac{\sin(\theta-\psi)}{\sin\psi}\cos\psi + \cos(\theta-\psi)\right]^2}{\cos(\theta-\psi)}$$

$$= \frac{(\sin(\theta-\psi)\cos\psi + \cos(\theta-\psi)\sin\psi)^2}{\sin^2\psi\cos(\theta-\psi)}$$

$$= \underbrace{\frac{\sin^2\theta}{\sin^2\psi}}_{A}\ \frac{1}{\cos(\theta-\psi)}$$

$$\cos^2(\theta-\psi) = 1-\sin^2(\theta-\psi)$$
$$= 1 - x^2\sin^2\psi$$
$$= 1 - x^2\underbrace{\sin^2\theta}_{A}$$

evaluate A:

$$\sin(\theta-\varphi) = x\sin\varphi$$
$$\sin\theta\cos\varphi - \cos\theta\sin\varphi = x\sin\varphi$$
$$\sin\theta\cos\varphi = (x+\cos\theta)\sin\varphi$$
$$\sin^2\theta\,(1-\sin^2\varphi) = (x+\cos\theta)^2\sin^2\varphi$$

$$\Rightarrow A = \frac{\sin^2\theta}{\sin^2\varphi} = 1+2x\cos\theta+x^2.$$

So
$$\cos^2(\theta-\varphi) = 1-\frac{x^2\sin^2\theta}{1+2x\cos\theta+x^2}.$$

$$= \frac{1}{1+2x\cos\theta+x^2}\left[1+2x\cos\theta+x^2\cos^2\theta\right]$$
$$= \frac{(1+x\cos\theta)^2}{1+2x\cos\theta+x^2}.$$

$\therefore$ $\cancel{RHS = \frac{1}{1+2x\cos\theta+x^2}}$

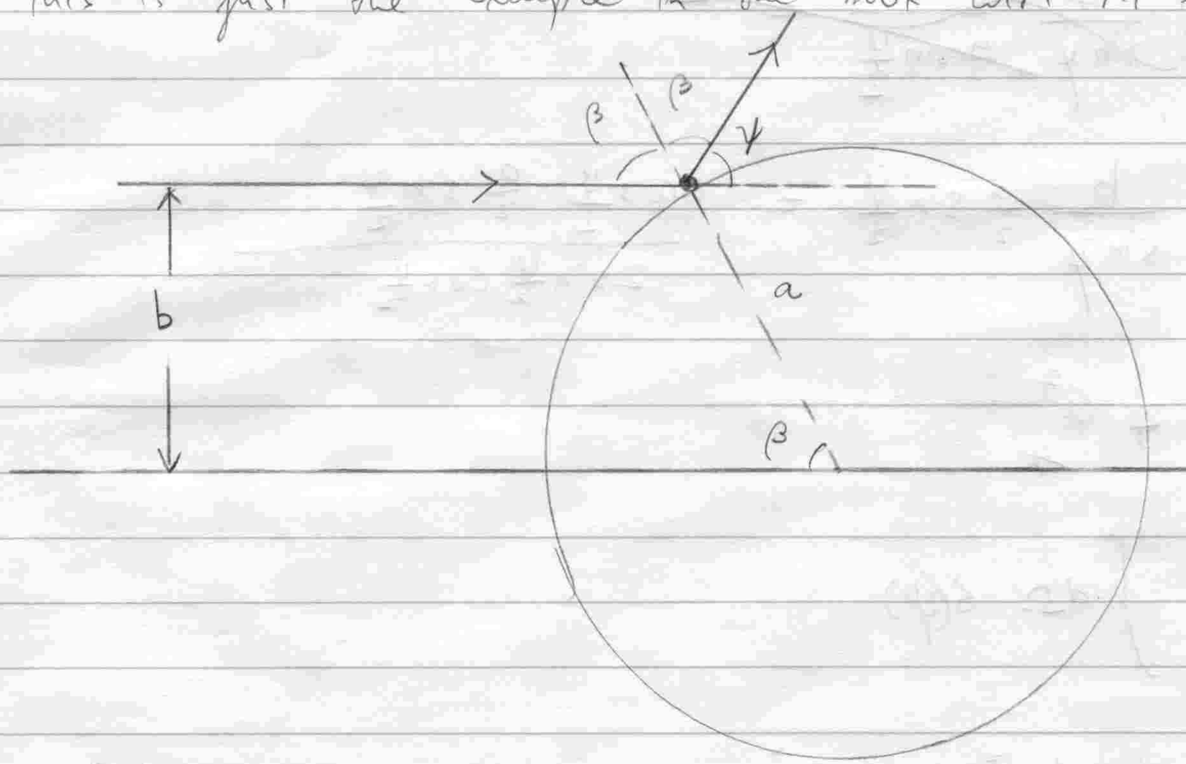
$$RHS = (1+2x\cos\theta+x^2)\,\frac{(1+2x\cos\theta+x^2)^{\frac{1}{2}}}{1+x\cos\theta}$$

or $$\frac{\sigma(\varphi)}{\sigma(\theta)} = \frac{(1+2x\cos\theta+x^2)^{\frac{3}{2}}}{1+x\cos\theta}.$$

#.

9-46. $U(r) = \begin{cases} 0, & r<a \\ \infty, & r>a \end{cases}$

In fact this is just the example in the book with $R_1 \to 0$:



$b = a \sin\beta$

$2\beta + \psi = \pi$

$$\Rightarrow b = a \sin\left(\frac{\pi}{2} - \frac{\psi}{2}\right) = a \cos\frac{\psi}{2}.$$

$$db = -\frac{a}{2} \sin\frac{\psi}{2}\, d\psi.$$

By conservation of particles ~~flux~~

$$2\pi b\, db = -\sigma(\psi) \underbrace{2\pi \sin\psi\, d\psi}_{d\Omega \text{ scattered}}$$

$$\sigma(\psi) = \left| -\frac{b}{\sin\psi}\frac{db}{d\psi} \right| = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

$$= \cancel{\left(\frac{b}{\sin\psi} \frac{1}{\frac{a}{2}\sin\frac{\psi}{2}} = 2 \right.}$$

$$= \frac{b}{\sin\psi}\,\frac{a}{2}\sin\frac{\psi}{2} = \frac{a\cos\frac{\psi}{2}\,\frac{a}{2}\sin\frac{\psi}{2}}{2\sin\frac{\psi}{2}\cos\frac{\psi}{2}}$$

$$= \frac{1}{4}a^2 \quad \#.$$

$$\sigma_t = \int d\Omega\, \sigma(\psi)$$

$$= 4\pi \frac{1}{4} a^2 = \pi a^2 \quad \#,$$

— geometric cross-sectional area of the sphere!

1-48. from Prob 1-45

$$\sigma(\psi) = \sigma(\theta) \frac{\sin^2\theta}{\sin^2\psi \cos(\theta-\psi)}$$

now $\chi = \frac{m_1}{m_2} >> 1$.

the scattering angle ψ must be small $\psi \simeq 0$

$$\cos(\theta-\psi) = \cos\theta \cos\psi + \sin\theta \sin\psi$$
$$\simeq \cos\theta$$
$$\sin\theta \simeq \sin(\theta-\psi) = \chi \sin\psi.$$

$$T_0' = \frac{m_2}{m_1+m_2} T_0 \simeq \frac{m_2}{m_1} T_0$$

therefore $\sigma(\psi) = \sigma(\theta) \frac{\chi^2 \sin^2\psi}{\cos\theta \sin^2\psi} = \frac{\chi^2}{\cos\theta}\sigma(\theta)$

$$= \frac{\chi^2}{\sqrt{1-\chi^2\sin^2\psi}}\sigma(\theta) \simeq \frac{\chi^2}{\sqrt{1-\chi^2\psi^2}}\sigma(\theta)$$

$$\sigma(\theta) = \frac{k^2}{(4T_0')^2} \frac{1}{\sin^4\frac{\theta}{2}} = \frac{k^2}{(4T_0')^2} \frac{4}{(1-\cos\theta)^2}.$$

$$= \left(\frac{m_1 k}{2m_2 T_0}\right)^2 \frac{1}{\left[1-\sqrt{1-\chi^2\sin^2\psi}\,\right]^2} \simeq \left(\frac{m_1 k}{2m_2 T_0}\right)^2 \frac{1}{\left[1-\sqrt{1-\chi^2\psi^2}\,\right]^2}$$

$$\therefore \sigma(\psi) = \left(\frac{m_1 k}{2m_2 T_0}\right)^2 \frac{\chi^2}{\left[1-\sqrt{1-\chi^2\psi^2}\,\right]^2 \sqrt{1-\chi^2\psi^2}} = \left(\frac{k\chi^2}{2T_0}\right)^2 \frac{1}{\left[1-\sqrt{1-\chi^2\psi^2}\,\right]^2 \sqrt{1-\chi^2\psi^2}}.$$

#.