Lagrangian Mechanics

An alternative method to obtain the Equation of Motion.

Advantages:

1) Rather than being forced to use an orthogonal coordinate system, we can choose the most convenient coordinate system for the given physical system, even non-orthogonal coordinates. This can simplify the analysis
   $\Longrightarrow$ Vector analysis is less important in Lagrangian Mechanics.

2) Lagrangian Mechanics often allows us to ignore the forces of constraint, which we usually do not care about anyway.

3) Lagrangian Mechanics gives us insight into conserved quantities like energy, momentum, and other quantities

4) Lagrangian Mechanics forms the basis for Hamiltonian Mechanics, which leads to Quantum Mechanics.

Disadvantages

1) Not well suited to systems which have dissipative (frictional) forces.

2) Perhaps we gain less physical intuition about why the system behaves the way it does, at least in terms of understanding the forces at work. But we do gain intuition about other aspects, such as the role of symmetry and conservation laws.

The Lagrangian Procedure:

We define the Lagrangian to be

$$\mathcal{L} \equiv T - U = \text{Kinetic Energy} - \text{Potential Energy.}$$

For a particle moving in a conservative potential (U) in 3 dimensions, we have, in Cartesian coordinates:

$$\mathcal{L} = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z)$$

We get an equation of motion for each coordinate by applying the Euler–Lagrange Equation:

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right), \quad \text{where } q = x, y, \text{ or } z \text{ and } \dot{q} = \dot{x}, \dot{y}, \text{ or } \dot{z}$$

So, for (x), we have

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow \boxed{-\frac{\partial U}{\partial x} = \frac{d}{dt}\left(m\dot{x}\right)}$$

similarly for (y) and (z):

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \Rightarrow -\frac{\partial U}{\partial y} = \frac{d}{dt}(m\dot{y})$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \Rightarrow -\frac{\partial U}{\partial z} = \frac{d}{dt}(m\dot{z})$$

In other words:

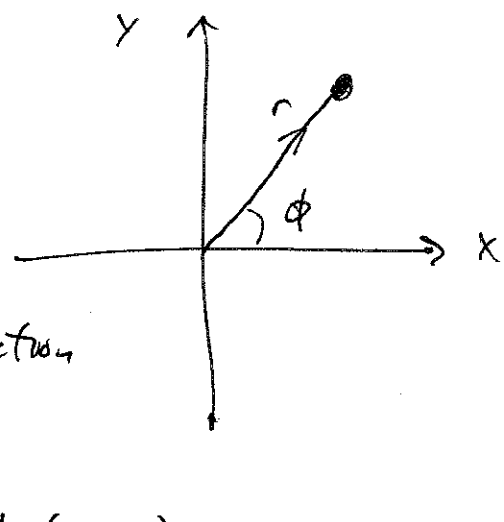
$$-\frac{\partial U}{\partial x} = m\ddot{x}, \quad -\frac{\partial U}{\partial y} = m\ddot{y}, \quad -\frac{\partial U}{\partial z} = m\ddot{z}$$

And since $\vec{F} = -\nabla U$, these are just the same as Newton's 2nd Law:

$$\vec{F} = m\vec{a}$$

<u>Ex: Two Dimensional Polar Coordinates.</u>

Let a particle move in the plane with position $\vec{r} = (r, \phi)$ and potential $U = U(r,\phi)$. Then



$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

KE due to (r) direction

KE due to ϕ direction

Then $\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi)$

<u>r Equation</u>

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$m r \dot{\phi}^2 \underbrace{- \frac{\partial U}{\partial r}}_{F_r} = \frac{d}{dt}\left(m \dot{r}\right) = m \ddot{r}$$

$$\therefore \quad \boxed{F_r = m\left(\ddot{r} - r\dot{\phi}^2\right)}$$

This is $F = ma$ for the radial coordinate in polar coordinates.

<u>ϕ Equation</u>

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}\left(m r^2 \dot{\phi}\right)$$

Recall that in Polar coordinates,

$$\nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi}, \quad \text{or} \quad F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$$

so $\quad -\dfrac{\partial U}{\partial \phi} = r F_\phi$

Therefore the ϕ Equation says

$$r F_\phi = \cancel{m}\frac{d}{dt}\left(m r^2 \dot{\phi}\right) = \frac{d}{dt}\left(\underbrace{\cancel{m}\, r\, m(r\dot{\phi})}_{\text{Angular Momentum}}\right)$$

$=$

or $rF_\phi = \frac{d}{dt}(L)$

↑ angular momentum about origin

And rF_ϕ is the torque about the origin, so

$$\text{torque} = \boxed{\Gamma = \frac{dL}{dt}}$$

By choosing polar coordinates, the ϕ equation naturally turned out to be the angular form of Newton's 2^{nd} Law. For this reason, we use the following terminology:

$$\frac{\partial \mathcal{L}}{\partial q} = \text{"Generalized Force"}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{"Generalized Momentum"}$$

The Generalized Force may have units of Newtons, or Newton·meters (torque), or some other units.

The Generalized Momentum may have units of kg·meters / second, or kg·(meter)^2/second (angular momentum), or some other units.

The exact form and units of the generalized force and momentum will depend on the coordinates we choose to use in our description of the system.

Conservation Laws – "Ignorable" or "Cyclic" coordinates

The Euler-Lagrange Equation says that

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \quad \text{for coordinate } q.$$

What happens when a particular coordinate q does not appear in the Lagrangian? Then

$$\frac{\partial \mathcal{L}}{\partial q} = 0 . \quad \text{so that} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$$

This equation says that $\frac{\partial \mathcal{L}}{\partial \dot{q}}$ will not change in time. In other words, it will be conserved.

When a coordinate does not appear in $\mathcal{L}$, we say that it is "ignorable" or "cyclic". Then the corresponding generalized momentum is conserved.

For example, for the particle moving in polar coordinates, if the potential U does not depend on ϕ, then we have

$$\mathcal{L} = \underbrace{\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}_{\text{the coordinate } \phi \text{ does not appear.}}$$

Then $\frac{\partial \mathcal{L}}{\partial \phi} = 0$, which means $\frac{d}{dt}(mr^2\dot{\phi}) = 0$

In other words, $mr^2\dot{\phi}$ is conserved, and this quantity is the angular momentum.

$$\boxed{\text{The absence of } \phi \text{ in the Lagrangian leads to angular momentum conservation.}}$$

Similarly, in cartesian coordinates, if $U(\vec{r})$ depends only on $\underline{z}$ $(U(z))$,

then $\mathcal{L} = \underbrace{\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(z)}_{}$

(x) and (y) do not appear.

Therefore we have 2 conserved generalized momenta:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = \text{constant} \quad (x\text{-momentum})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} = \text{constant} \quad (y\text{-momentum})$$

The absence of (x) & (y) in the Lagrangian leads to conservation of momentum in the x & y directions. In the (z) direction, however, we have

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\ddot{z}$$

$$-\frac{\partial U}{\partial z} = F_z \quad \text{or} \quad \boxed{F_z = m\ddot{z}}$$

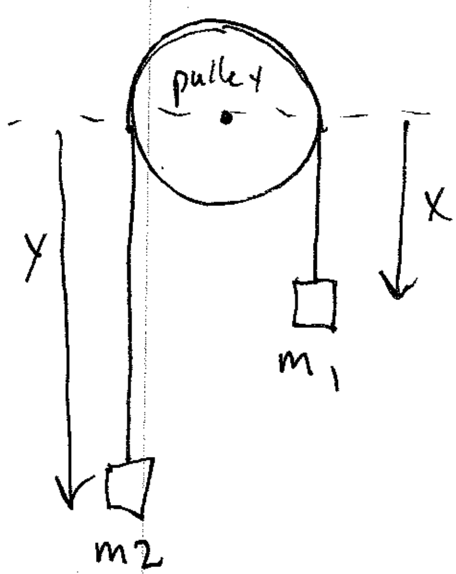
By identifying conserved quantities we can reduce the complexity of the system. So it is advantageous to choose coordinates such that the resulting Lagrangian depends on the smallest number of coordinates.

> We always try to choose coordinates such that ~~as many~~ we maximize the number of ignorable coordinates.

Constrained systems

We will prove later that the Lagrangian Formalism still works when the system is subject to forces of constraint.

Example of a constrained system: Atwood Machine



(x) and (y) are not independent, they are subject to the constraint

$$y + x = \text{constant}$$

because the length of the rope is constant. Therefore

$$y = -x + \text{constant}$$
$$\dot{y} = -\dot{x}.$$

The kinetic energy is

$$\dot{y} = -\dot{x}$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2$$

Potential Energy: $U = -m_1 g x - m_2 g y$

$$= -m_1 g x - m_2 g(-x + \text{constant})$$

$$= -(m_1 - m_2) g x + \text{constant}$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{x}^2 + (m_1-m_2)gx + (\text{drop the unnecessary constant})$$

Equation of Motion: only one coordinate:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$(m_1-m_2)g = \frac{d}{dt}(m_1+m_2)\dot{x} = (m_1+m_2)\ddot{x}$$

or $\boxed{\ddot{x} = \left(\frac{m_1-m_2}{m_1+m_2}\right)g}$ Eq. of Motion.

Historically the Atwood Machine was used to measure g, because one can choose m_1 and m_2 are very close to each other.

Then the acceleration ($\ddot{x}$) is very small and easier to measure

Newtonian Method:

Each mass also experiences a force due to tension in the rope. This tension is the same for both masses.

Mass 1: $m_1 g - F_t = m_1 \ddot{x}$

Mass 2: $m_2 g - F_t = m_2 \ddot{y} = -m_2 \ddot{x}$

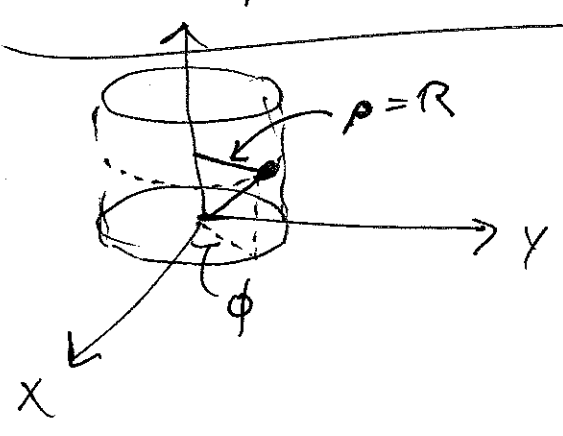
$$\Rightarrow F_t - m_2 g = m_2 \ddot{x}$$

Add Equations to eliminate F_t:

$$m_1 g - m_2 g = m_1 \ddot{x} + m_2 \ddot{x}$$

$$\boxed{\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} g}$$ Eq. of Motion.

Another example of a constrained system: Particle constrained on a cylinder ~~with~~ plus a spring force.



R = radius of the cylinder.
~~Ch~~ Choose ϕ and z as the coordinates.
(~~ρ~~ ρ is fixed at R.)

Let's assume there is a spring force Directed toward the origin and proportional to the distance to the origin:

$$\vec{F} = -k\vec{r}, \quad \text{where } \vec{r} = (x, y, z)$$

The Kinetic Energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\Big(\underbrace{(R\dot{\phi})}_{\substack{\text{velocity} \\ \text{in the } \hat{\phi} \text{ direction}}}{}^2 + \dot{z}^2\Big) = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right)$$

The Potential Energy is

$$U = \frac{1}{2}k\underbrace{\left(R^2 + z^2\right)}_{\substack{\text{Distance squared} \\ \text{to the origin.}}}$$

So $\mathcal{L} = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$

Two Coordinates $\Rightarrow$ Two Equations of Motion

<u>z Equation:</u> $\dfrac{\partial \mathcal{L}}{\partial z} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{z}}$

$$\boxed{-kz = m\ddot{z}} \Rightarrow \text{Simple Harmonic Motion in } (z).$$

<u>ϕ Equation:</u> $\underbrace{\dfrac{\partial \mathcal{L}}{\partial \phi}}_{\emptyset} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}$

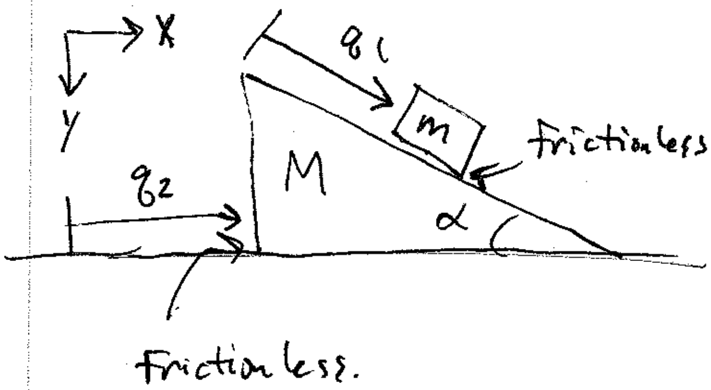
$\emptyset$, because ϕ is <u>ignorable</u>.

so $\underbrace{mR^2\dot{\phi}}_{\text{angular momentum about the origin.}} = \text{constant} \Rightarrow \boxed{L = \text{constant}}$

$$\Downarrow$$

$$\boxed{\dot{\phi} = \text{constant}}$$

Block sliding on a wedge (Frictionless)



How long does it take the block to reach the bottom? No friction between block and wedge, and no friction between wedge and table.

As the block slides down, the wedge moves to the left.

KE of wedge: $\frac{1}{2}M\dot{q}_2^2$

KE of block: $\vec{v} = (v_x, v_y) = (\underbrace{\dot{q}_1\cos\alpha}_{\text{x velocity of block on wedge}} + \underbrace{\dot{q}_2}_{\text{x velocity of wedge on table.}}, \dot{q}_1 \sin\alpha)$

$$T_m = \frac{1}{2}m(v_x^2 + v_y^2)$$
$$= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha)$$

Potential Energy: $U = -mgq_1\sin\alpha$

Lagrangian: $\mathcal{L} = T - U =$
$$= \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha)$$
$$+ mgq_1\sin\alpha$$

q_2 Equation: $\dfrac{\partial \mathcal{L}}{\partial q_2} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_2} = \dfrac{d}{dt}\left(M\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1\cos\alpha)\right)$

$\Downarrow$

$\emptyset$

or $M\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1\cos\alpha)$ = generalized momentum = constant.

q_1 Equation: $\dfrac{\partial \mathcal{L}}{\partial q_1} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_1}$

$$mg\sin\alpha = \frac{d}{dt}\left(m(\dot{q}_1 + \dot{q}_2\cos\alpha)\right) = m\ddot{q}_1 + \ddot{q}_2\cos\alpha$$

We can differentiate the q_2 Equation to get

$$M\ddot{q}_2 = -m(\ddot{q}_2 + \ddot{q}_1\cos\alpha)$$

$$\ddot{q}_2 = -\frac{m\ddot{q}_1\cos\alpha}{M+m}$$

Eliminate $\ddot{q}_2$ from the q_1 Equation:

$$mg\sin\alpha = m\ddot{q}_1 - \frac{m\cos^2\alpha\,\ddot{q}_1}{M+m}$$

$$\boxed{\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{M+m}}}$$

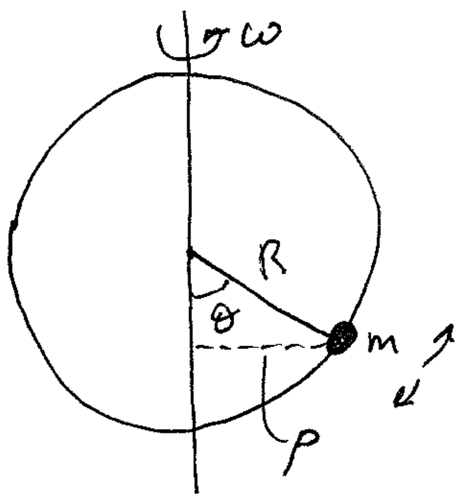
$\Leftarrow$ This is a constant acceleration down the wedge.

The time to reach the bottom will satisfy

length of ramp $\longrightarrow$ $l = \frac{1}{2}\ddot{q}_1 t_{bottom}^2 \Longrightarrow \boxed{t_{bottom} = \sqrt{\dfrac{2l}{\ddot{q}_1}}}$

Bead ~~Spinning~~ on a Spinning Hoop

A frictionless hoop has a bead which is free to move. The entire hoop rotates about one of its diameters at frequency (ω). What is the equation of motion for the bead?



The bead is free to move up and down the hoop with angle θ. It also rotates in the plane around the axis of the hoop.

Tangential velocity: $R\dot{\theta}$

Normal velocity (due to hoop rotation): $p\omega = (R\sin\theta)\omega$

Kinetic Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta)$

Potential Energy: $U = mgR(1-\cos\theta)$

Lagrangian: $\mathcal{L} = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR(1-\cos\theta)$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$$

$$mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta = \frac{d}{dt}(mR^2\dot{\theta}) = mR^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta}$$ Equation of motion.

Let's find the Equilibrium points for θ, locations (call them θ_0) where the acceleration $\ddot{\theta}$ is zero. At these locations, a bead placed there at rest will remain there.

Set $\ddot{\theta}$ equal to zero:

$$\left(\omega^2 \cos\theta_0 - \frac{g}{R}\right) \sin\theta_0 = 0$$

so $\theta_0 = 0$ or π will both be equilibrium points.

But also $\cos\theta_0 = \dfrac{g}{\omega^2 R}$ can work.

This condition can be satisfied when $\dfrac{g}{\omega^2 R} \leq 1$

or $\omega^2 \geq \dfrac{g}{R}$

Then there are two additional equilibrium points.

$$\theta_0 = \pm \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$$

On the other hand, if $\omega^2 < g/R$ (slow rotation) only $\theta = 0$ and $\theta = \pi$ will be equilibrium points.

Stability of the Equilibria

For $\theta_0 = 0$, we can approximate the Eq. of Motion with $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ so

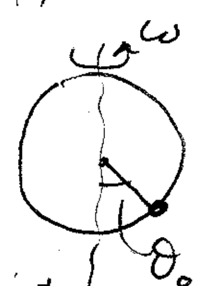
$$\cancel{\ddot{\theta} \approx (\omega^2 \cos\theta - \tfrac{g}{R})\theta}$$

$$\ddot{\theta} \approx \left(\omega^2 - \frac{g}{R}\right)\theta \qquad (\theta \text{ near } 0)$$

For $\omega^2 < g/R$, $(\omega^2 - g/R)$ is negative and constant, and we have the simple harmonic oscillator Eq. of Motion. This is a stable equilibrium, oscillation frequency becomes $= \sqrt{g/R - \omega^2}$

But if $\omega^2 > g/R$, then $(\omega^2 - g/R)$ is positive, and rather than restoring the equilibrium, the displacement further reinforces the ~~equilibrium~~ motion. This is unstable.

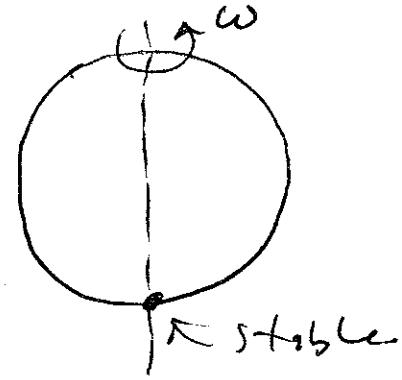
What about the two equilibria which appear when $\omega^2 > g/R$: then



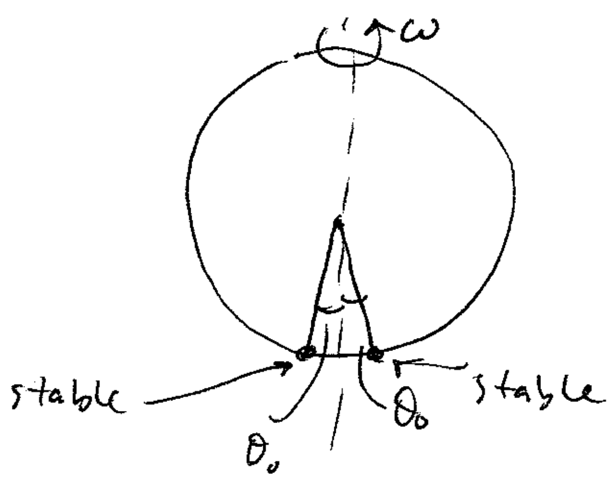
$$\ddot{\theta} = \underbrace{\left(\omega^2 \cos\theta - g/R\right)}_{\text{zero at the equilibrium point.}} \sin\theta$$

When θ increases, $\sin\theta$ remains positive, but $\cos\theta$ decreases, so the factor in parentheses becomes negative. This is a stable equilibrium.

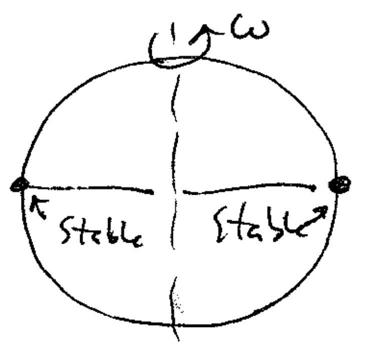
So, when the hoop rotates slowly, there is only one stable equilibrium, with the bead at the bottom:



But if the speed increases, with $\omega^2 \geq g/R$, then the bottom position becomes unstable, but two stable equilibria appear on either side of the hoop:



As $\omega \to \infty$, the stable equilibria $\to \pm 90°$:

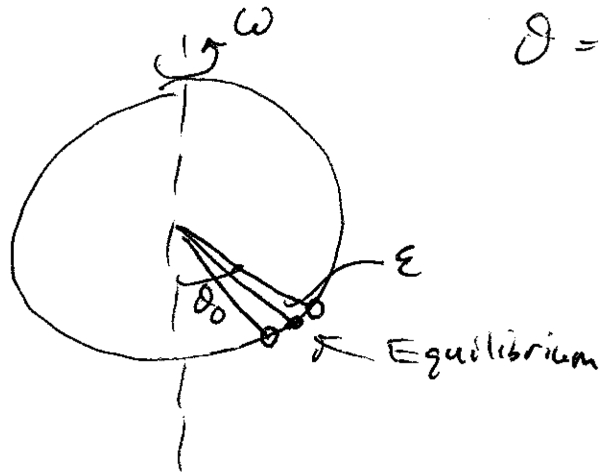


Oscillation frequency; for $\omega^2 > g/R$

θ_0 satisfies $\omega^2 \cos\theta_0 - g/R = 0$.

Now let's allow the bead to move about θ_0 by angle ε:

$$\theta = \theta_0 + \varepsilon$$



Then $\cos\theta = \cos(\theta_0+\varepsilon) \approx \cos\theta_0 - \varepsilon\sin\theta_0$

and $\sin\theta = \sin(\theta_0+\varepsilon) \approx \sin\theta_0 + \varepsilon\cos\theta_0$

Then the Eq. of Motion becomes

$$\ddot{\theta} = \left(\omega^2\cos(\theta_0+\varepsilon) - g/R\right)\sin(\theta_0+\varepsilon)$$

$$\approx \left(\omega^2\cos\theta_0 - \varepsilon\omega^2\sin\theta_0 - g/R\right)\left(\sin\theta_0 + \varepsilon\cos\theta_0\right)$$

cancel by definition of θ_0

Then drop the ε^2 terms: (and note that $\ddot{\theta} = \ddot{\varepsilon}$)

$$\boxed{\ddot{\varepsilon} \approx -\omega^2\sin^2\theta_0\,\varepsilon}$$

And the oscillation frequency is

Ω = oscillation frequency = $\omega \sin \vartheta_0$