

Motion of a charge in a uniform magnetic field

$$\vec{F} = q \vec{v} \times \vec{B} = m \dot{\vec{v}}$$

Let \vec{B} be constant and in the \hat{z} direction:

$$\vec{B} = (0, 0, B)$$

Also $\vec{v} = (v_x, v_y, v_z)$

Then $\vec{v} \times \vec{B} = (v_y B, -v_x B, 0)$

So the three components of $\vec{F} = m \dot{\vec{a}}$ are

$$m \dot{v}_x = q B v_y$$

$$m \dot{v}_y = -q B v_x$$

$$m \dot{v}_z = 0$$

$$\implies v_z = \text{constant} = v_{z0}$$

Simplify x & y equations by defining

$$\omega = \text{"cyclotron frequency"} \equiv \frac{qB}{m}$$

Then $\left[\begin{array}{l} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega v_x \end{array} \right]$ Coupled differential Equations

These can be easily solved using a trick with complex numbers.

Define $\eta \equiv v_x + i v_y$

Then $\dot{\eta} = \dot{v}_x + i \dot{v}_y$
 $= \omega v_y + i(-\omega v_x)$
 $= -i\omega (i v_y + v_x)$
 $= -i\omega (v_x + i v_y)$
 $= -i\omega \eta$

by the equations of motion.

$$\boxed{\dot{\eta} = -i\omega \eta} \quad \text{Equations of Motion}$$

Solution: $\boxed{\eta = A e^{-i\omega t}}$

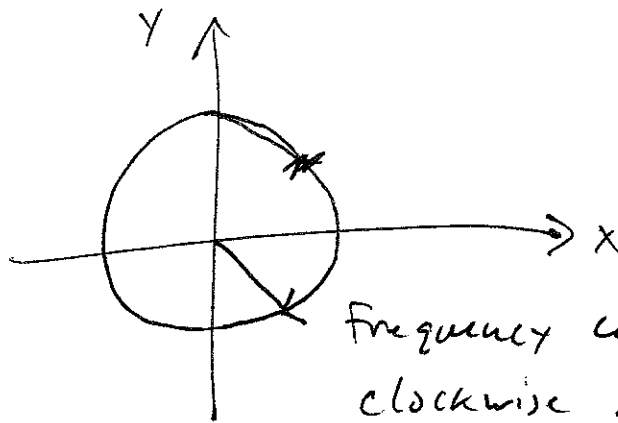
We can integrate once more to get $x(t)$ & $y(t)$:

Define: $\xi = x + i y$. Then $\dot{\xi} = \eta$, so

$$\xi = \int \eta dt = \int A e^{-i\omega t} dt$$

$$= \frac{iA}{\omega} e^{-i\omega t} + (\text{complex constant})$$

So ξ rotates in the xy plane with frequency ω :



frequency ω ,
clockwise motion for $+q$

because $\xi \sim e^{-i\omega t}$
 \uparrow
 (-) sign
 means
 clockwise.

Meanwhile z increases uniformly:

$$\boxed{z(t) = z_0 + v_{0z}t} \Rightarrow \text{helical motion}$$

The complex constant of integration can be chosen to be zero by choosing $(x, y) = (0, 0)$ to be the center of the circular motion.

The initial velocities v_{x0} and v_{y0} affect the magnitude and phase of A as well as the center of the motion.

$$\text{Radius of the orbit} = r = \frac{v}{\omega} = \frac{v}{qB/m} = \frac{mv}{qB} = \frac{p}{qB}$$

Conservation of Momentum

By Newton's 3rd law we can infer that

$$\frac{d\vec{P}_{\text{total}}}{dt} = \vec{F}_{\text{ext}}$$

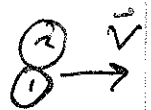
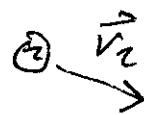
because all internal forces cancel.

If $\vec{F}_{\text{ext}} = 0$, then total momentum \vec{P}_{total} is conserved: $\vec{P}_{\text{total}} = \text{constant}$

~~Ex:~~ Ex: Inelastic collision

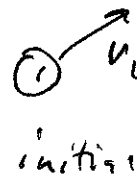
Let 2 particles collide and stick together:

$$\vec{P}_{\text{initial}} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$



$$\vec{P}_{\text{final}} = m_1 \vec{v} + m_2 \vec{v} = (m_1 + m_2) \vec{v}$$

Common
final
velocity



Final

$$\vec{P}_{\text{initial}} = \vec{P}_{\text{final}}$$

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = (m_1 + m_2) \vec{v}$$

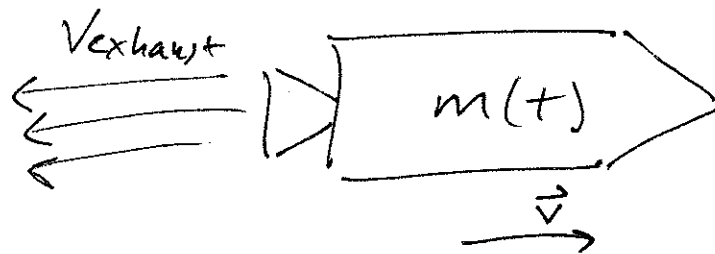
$$\vec{v} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

Special case: $\vec{v}_2 = \phi$. Then

$$\vec{v} = \left(\frac{m_1}{m_1 + m_2} \right) \vec{v}_1$$

Rocket Motion

Rockets in free space accelerate by jettisoning mass. Conservation of momentum gives them a boost. (The jettisoned mass is in the form of exhaust gas.)



For this case the mass of the rocket depends on time:

$$m(t)$$

And $\dot{m}(t)$ is negative, because the rocket is becoming lighter.

At time t , $P(t) = m(t)v(t)$

At time $t+dt$, $P(t+dt) = (m(t) + \underset{\substack{\uparrow \\ \text{change in mass,} \\ \text{dm is negative.}}}{dm})(v(t) + dv)$

The exhaust ejected in time dt has mass (dm) and velocity $v(t) - v_{ex}$ relative to an ~~at~~ inertial coordinate system at rest.

So the total momentum is

$$P(t+dt) = \underbrace{(m(t) + dm)}_{\text{rocket}} (v(t) + dv) - \underbrace{dm (v(t) - v_{ex})}_{\text{exhaust}}$$

$$= m(t)v(t) + m(t)dv + \cancel{v(t)dm} + \underbrace{dm dv}_{\text{ignore term being very small}} - \cancel{v(t)dm} + v_{ex}dm$$

$$= \underbrace{m(t)v(t)}_{P(t)} + m(t)dv + v_{ex}dm$$

$$\text{So } dP = P(t+dt) - P(t) = m(t)dv + v_{ex}dm$$

Since momentum is conserved, $dP = 0$:

$$m(t)dv + v_{ex}dm = 0$$

$$m(t)dv = -dm v_{ex}$$

$$\text{or } m(t) \frac{dv}{dt} = -\frac{dm}{dt} v_{ex}$$

$$\boxed{m(t) \dot{v} = -\dot{m}(t) v_{ex}}$$

Eq. of Motion

This looks like an ordinary Newton's 2nd Law Eq. of Motion where $-\dot{m}(t)v_{ex}$ plays the role of the force. we call it the thrust.

$$\boxed{\text{Thrust} \equiv -\dot{m}(t) v_{ex}}$$

We can solve the Eq. of motion by separation of variables:

$$dv = \frac{-v_{ex} dm}{m(t)}$$

$$v_{\text{Final}}^{(+)} - v_0 = -v_{ex} \int_{m_0}^{m(t)} \frac{dm'}{m'(t)}$$

$$= -v_{ex} \ln\left(\frac{m(t)}{m_0}\right)$$

$$\boxed{v_{\text{Final}}^{(+)} - v_0 = v_{ex} \ln\left(\frac{m_0}{m(t)}\right)}$$

Center-of-Mass

We define $\vec{R} \equiv$ center of mass of a system of particles

$$\equiv \frac{1}{M} \sum_{\alpha} m_{\alpha} \vec{r}_{\alpha} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_N \vec{r}_N}{M}$$

where $M \equiv \sum_{\alpha} m_{\alpha} =$ total mass.

Three components: $R_x = \frac{1}{M} \sum_{\alpha} m_{\alpha} r_{\alpha x}$, $R_z = \frac{1}{M} \sum_{\alpha} m_{\alpha} r_{\alpha z}$

$$R_y = \frac{1}{M} \sum_{\alpha} m_{\alpha} r_{\alpha y}$$

The total momentum can be written in terms of the CM:

$$\vec{P}_{\text{total}} = \sum_{\alpha} \vec{p}_{\alpha} = \sum_{\alpha} m_{\alpha} \dot{\vec{r}}_{\alpha} = M \dot{\vec{R}}$$

Taking the derivative we have

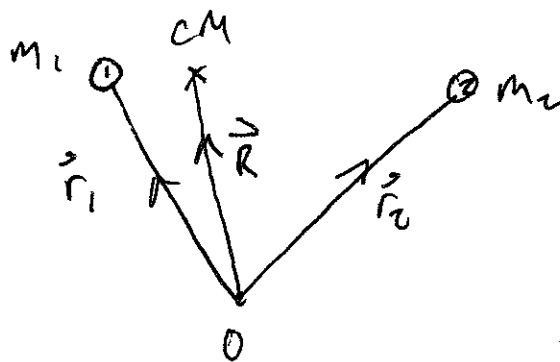
$$\dot{\vec{P}}_{\text{total}} = M \ddot{\vec{R}}$$

↑

\vec{F}_{ext} by Newton's 2nd Law

$$\boxed{\vec{F}_{\text{ext}} = M \ddot{\vec{R}}}$$

The CM moves like a single particle subject to a single force (\vec{F}_{ext}).



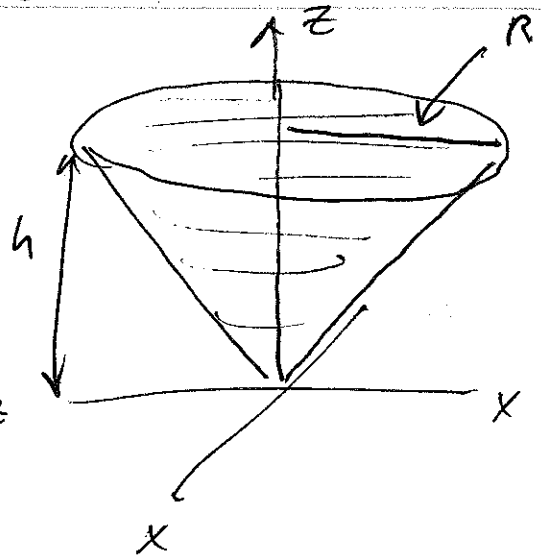
For a continuous mass distribution,

$$\vec{R} \equiv \frac{1}{M} \int_{\text{volume}} \rho \vec{r} dV, \quad \text{where } \rho(x, y, z) \text{ is the mass density.}$$

Ex: CM of a cone:

uniform solid cone
(density = $\rho = \text{constant}$)

$$R_x = \frac{1}{M} \int \rho(x, y, z) x \, dx \, dy \, dz$$



By symmetry, every point

$$(\rho(x, y, z) x)$$

is cancelled by a point $(\rho(-x, y, z)(-x))$

$$\text{So } R_x = 0.$$

Similarly for R_y : $R_y = 0$

For R_z , there is no cancellation:

$$R_z = \frac{\rho}{M} \int z \, dx \, dy \, dz$$

The integrals over $dx \, dy$ give the area of a circle: πr^2 .

$$\text{But } r \text{ is related to } z: \quad r = \frac{Rz}{h}$$

$$\text{So } R_z = \frac{\rho}{M} \int z \pi \left(\frac{Rz}{h}\right)^2 dz$$

$$= \frac{\rho R^2 \pi}{M h^2} \int_0^h z^3 dz = \frac{\rho R^2 \pi}{\frac{1}{4} M h^2} \left(\frac{1}{4} h^4\right)$$

The volume of the cone is

$$V = \frac{1}{3} \pi R^2 h$$

so the mass is $M = \frac{\rho \pi R^2 h}{3}$

Therefore $R_z = \frac{3}{4} h$

Angular Momentum (Single Particle)

We define $\vec{l} \equiv \vec{r} \times \vec{p}$, \vec{r} = position vector
 \vec{p} = momentum.

Its time rate change is

$$\dot{\vec{l}} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}$$

\uparrow $\vec{p} = m\dot{\vec{r}}$
 \uparrow \vec{F}_{net} by Newton's 2nd Law

so $\vec{r} \times m\dot{\vec{r}} = 0$ (parallel vectors have zero cross product)

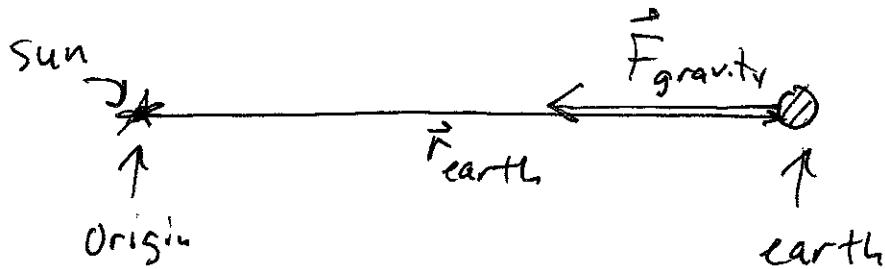
$$\therefore \dot{\vec{l}} = \vec{r} \times \vec{F}$$

Define $\vec{r} \times \vec{F} \equiv \vec{\tau}$ = "torque"

The $\boxed{\dot{\vec{l}} = \vec{\tau}}$ Newton's 2nd Law in angular form

Torque and Angular Momentum ~~are~~ behave somewhat differently than force and linear momentum. For example, changing the origin changes the torque:

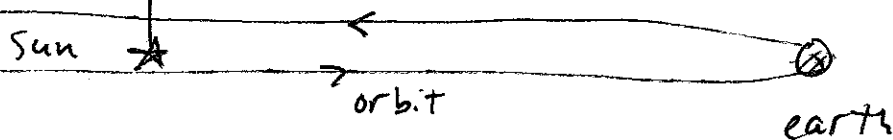
Ex: Central Force (sun-earth system)



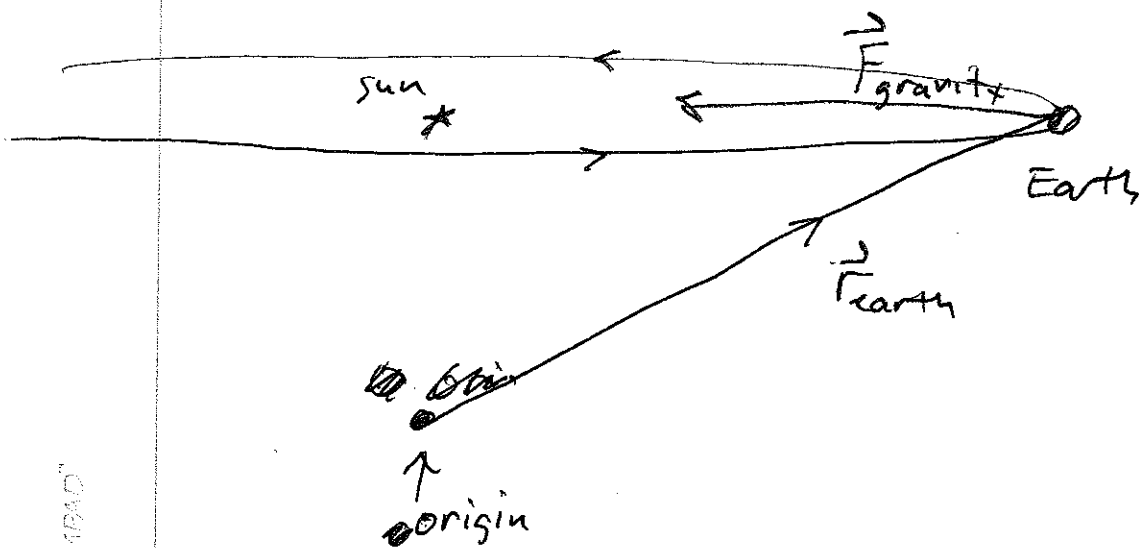
With the sun as the origin,

$\vec{\Gamma} = \vec{r} \times \vec{F}_{gravity} = \emptyset$ because \vec{r} and \vec{F} are parallel. With this choice of origin, angular momentum is conserved:

$$\vec{l}_{earth} = \text{constant}$$

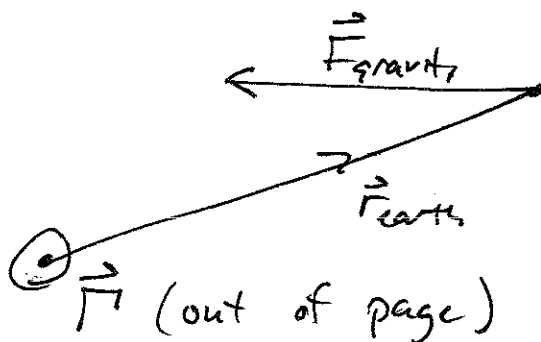


But suppose we choose an origin below the sun, and below the plane of the earth's motion:

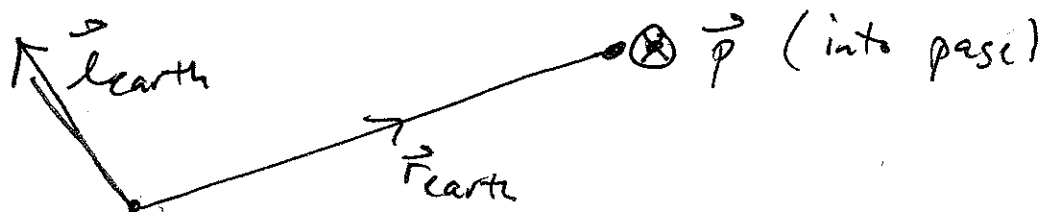


Answer

Now the torque is non-zero and points out of the page:



The angular momentum vector is

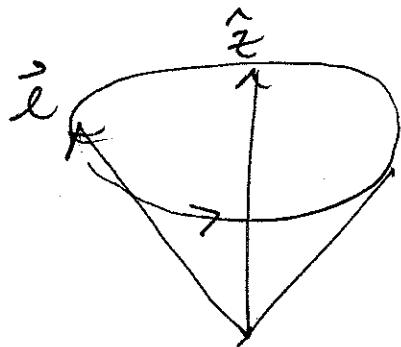


According to Newton's 2nd Law,

$$\vec{\tau} = \dot{\vec{L}}, \text{ so } \vec{L} \text{ must be changing}$$

in response to this torque

How is \vec{L} changing? It is rotating around the \hat{z} axis on a cone:



\vec{L} always remains \perp to both \vec{r} and \vec{p} .

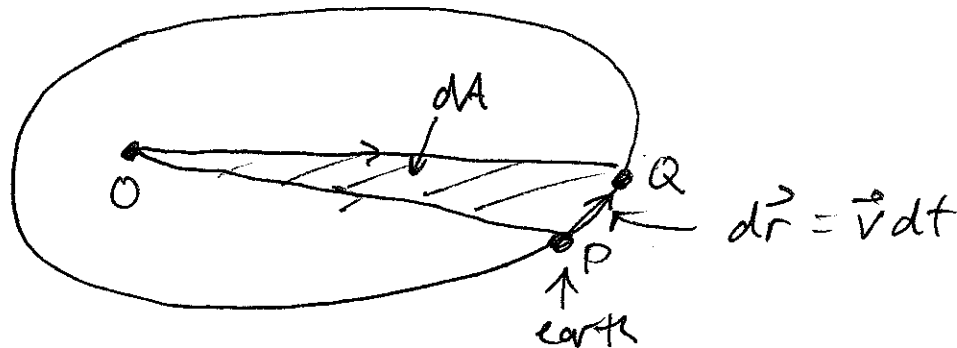
So with the origin at the sun, the situation is simple: $\vec{\Gamma} = 0$, and $\vec{L} = \text{constant}$. But a different choice of origin gives $\vec{\Gamma} = \text{non-zero}$ and \vec{L} changing in time.

Kepler's 2nd Law:

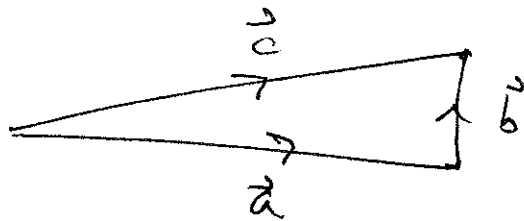
"As each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal area in equal time."

This is a consequence of angular momentum conservation (taking the sun as the origin.)

We can see this as follows.



From homework # 1.18, we know that the area of a triangle can be calculated by taking the cross product of any two of its sides.



$$\text{Area} = \frac{1}{2} |\vec{a} \times \vec{b}| \text{ for example.}$$

So the area OPQ is

$$dA = \frac{1}{2} |\vec{r} \times \vec{v} dt|$$

Replace \vec{v} by \vec{p}/m and divide by dt :

$$\frac{dA}{dt} = \frac{1}{2m} |\vec{r} \times \vec{p}| = \frac{1}{2m} |\vec{l}|$$

But with our choice of origin, $|\vec{l}| = \text{constant}$,

so

$$\boxed{\frac{dA}{dt} = \text{constant}}$$

This is Kepler's
2nd Law.

Alternatively, $l = mr^2\omega$. The geometrical rate is

$$\frac{dA}{dt} = \frac{1}{2} r^2\omega$$

So $\frac{dA}{dt} = \text{constant}$ if $l = \text{constant}$.

Total Angular Momentum of a system of particles:

Define
$$\vec{L} \equiv \sum_{\alpha} \vec{l}_{\alpha} = \sum_{\alpha} \vec{r}_{\alpha} \times \vec{p}_{\alpha}$$

It is a simple ~~easy~~ exercise to show that if all internal forces between particles obey Newton's 3rd Law (equal & opposite), then

$$\dot{\vec{L}} = \vec{\tau}_{\text{ext}}$$

where $\vec{\tau}_{\text{ext}}$ are the total torques due to external forces. In particular, if $\vec{\tau}_{\text{ext}} = \phi$, then total angular momentum is conserved.

Moment of Inertia

For a fixed axis, which is not allowed to wobble, the moment of inertia is defined as:

$$I \equiv \sum_{\alpha} m_{\alpha} p_{\alpha}^2,$$

where $p_{\alpha} \equiv$ distance from particle (α) to the axis.

For a continuous mass distribution,

$$I \equiv \int \rho r_{\perp}^2 dV$$

$\uparrow \quad \uparrow$ r_{\perp} = distance to axis of rotation.
 $\rho = \rho(x, y, z)$ = density function

Some simple items:

~~Uniform disk~~ uniform disk: $I = \frac{1}{2} MR^2$

$\uparrow \uparrow$ radius
 total
 mass

Solid sphere rotating about one axis: $I = \frac{2}{5} MR^2$

The usefulness of the moment of inertia is that you can quickly calculate the \hat{z} -component of \vec{L} (for a fixed rotation axis) by multiplying in the \hat{z} direction

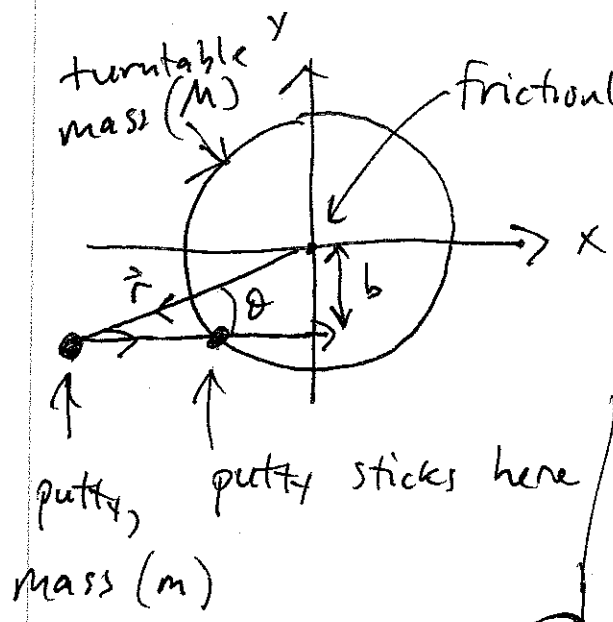
$$L_z = I\omega$$

One can then determine the time rate change by applying Newton's 2nd Law:

$$I\dot{\omega} = \tau_z$$

Angular collision problems can be solved by applying conservation of L .

Ex: Putty thrown at a ~~slow~~ stationary turn table:



Initial Angular Momentum:

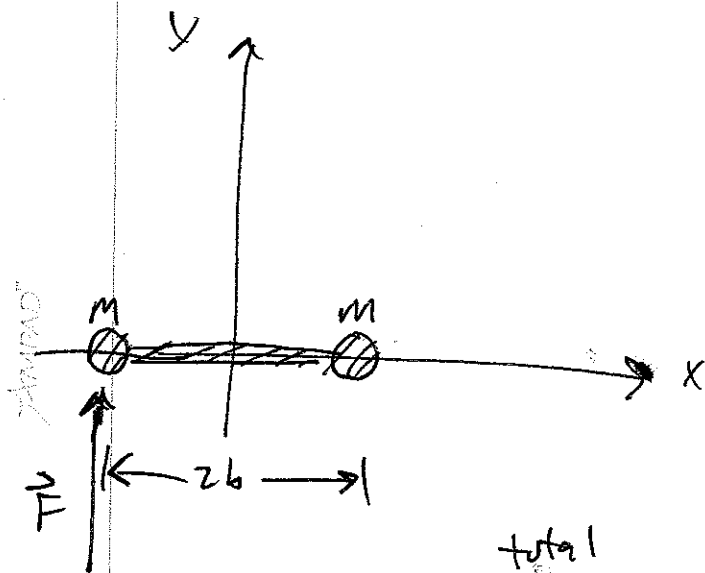
$$L_z = r(mv) \sin \theta = mvb$$

After collision, the moment of inertia is

$$I = \underbrace{mR^2}_{\substack{\uparrow \\ \text{I for} \\ \text{the putty}}} + \underbrace{\frac{M}{2}R^2}_{\substack{\uparrow \\ \text{I for the} \\ \text{turntable}}} = \left(m + \frac{M}{2}\right)R^2$$

$$L_z^{\text{Final}} = \left(m + \frac{M}{2}\right)R^2 \omega = mvb$$

$$\omega = \frac{m}{\left(m + \frac{M}{2}\right)} \frac{vb}{R^2}$$

Ex: Struck Dumbbell

A force strikes the left dumbbell for a short time Δt .

$$M = 2m$$

The change in ^{total} linear momentum is

$$\Delta \vec{P} = \vec{F} \Delta t$$

Starting with $\vec{P}_{\text{initial}} = \vec{0}$, we have

$$\vec{P}_{\text{final}} = M \vec{V} = \vec{F} \Delta t \quad \Rightarrow \quad \vec{V} = \vec{V}_{\text{cm}} = \left(\frac{\vec{F} \Delta t}{M} \right)$$

total velocity of CM mass

The change in total angular momentum about the origin is due to the torque

$$|\tau_{\text{ext}}| = Fb$$

$$\Delta L = |\tau_{\text{ext}}| \Delta t = Fb(\Delta t)$$

$$L_{\text{initial}} = \vec{0}, \quad \text{so} \quad \Delta L = L_{\text{final}} = I\omega$$

$$I = 2mb^2 = Mb^2, \text{ so}$$

$$I\omega = Fb\Delta t$$

$$(Mb^2)\omega = Fb\Delta t$$

$$\omega = \frac{F\Delta t}{Mb} = \text{angular velocity}$$

The left mass has velocity:

$$v_{\text{left}} = v_{\text{cm}} + \omega b = \frac{F(\Delta t)}{M} + \left(\frac{F(\Delta t)}{Mb}\right)b$$

$$= \frac{2F(\Delta t)}{M}$$

$$= \frac{F(\Delta t)}{M/2}$$

$$= \frac{F\Delta t}{m}$$

The right mass initially has velocity

$$v_{\text{right}} = v_{\text{cm}} - \omega b = 0$$

The CM moves vertically up the y-axis while the dumbbell rotates.

Work & Energy

We define $KE \equiv T \equiv \frac{1}{2}mv^2$ for a single particle.

The time derivative is

$$\begin{aligned} \frac{dT}{dt} &= \frac{d}{dt} \left(\frac{1}{2} m \vec{v} \cdot \vec{v} \right) \\ &= \frac{1}{2} m \dot{\vec{v}} \cdot \vec{v} + \frac{1}{2} m \vec{v} \cdot \dot{\vec{v}} \end{aligned}$$

$$\frac{dT}{dt} = m \dot{\vec{v}} \cdot \vec{v}$$

From Newton's 2nd law, $m\dot{\vec{v}} = \vec{p} = \vec{F}_{ext}$, so

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

or

$$\boxed{dT = \vec{F} \cdot d\vec{r}}$$

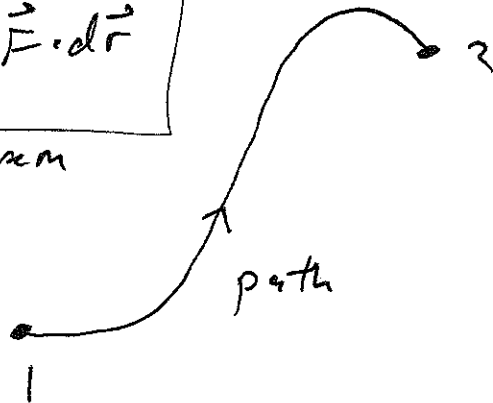
We define $\vec{F} \cdot d\vec{r}$ to be the differential work (dW) done by force $\vec{F} = \underline{\underline{\quad}}$

$$dW \equiv \vec{F} \cdot d\vec{r}$$

The total work done is found by integrating dW over the entire path:

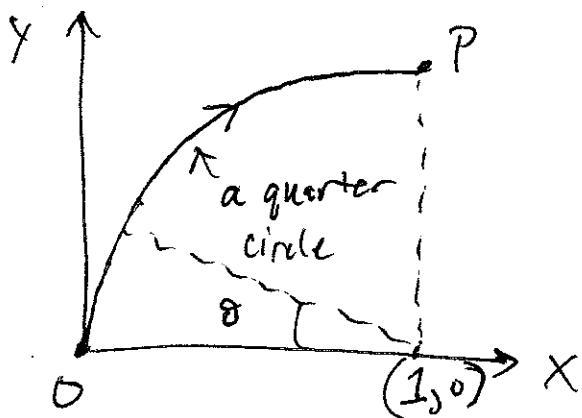
$$\Delta T = T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r}$$

Work-energy theorem



The Work ~~Energy~~ Energy Theorem follows from Newton's 2nd Law and our definitions of T & W.

Example Work calculation:



Path: $\vec{r} = (1 - \cos \theta, \sin \theta)$
 $d\vec{r} = (\sin \theta, \cos \theta) d\theta$
 Force Law: $\vec{F} = (y, 2x)$
 $= y\hat{x} + 2x\hat{y}$

Notice that $d\vec{r} = (\underbrace{\sin \theta \hat{x} + \cos \theta \hat{y}}_{\text{direction}}) \underbrace{d\theta}_{\text{magnitude}}$

Line integral along path:

$$W = \int_0^P (y\hat{x} + 2x\hat{y}) \cdot (\sin \theta \hat{x} + \cos \theta \hat{y}) d\theta$$

$$= \int_0^{\pi/2} (\sin \theta \hat{x} + 2(1 - \cos \theta)\hat{y}) \cdot (\sin \theta \hat{x} + \cos \theta \hat{y}) d\theta$$

$$= \int_0^{\pi/2} (\sin^2 \theta + 2(1 - \cos \theta)\cos \theta) d\theta = \boxed{2 - \pi/4}$$

Conservative Forces

If the work performed by a force ~~is~~ ~~that~~ on a particle travelling from point ① to point ② depends only on:

(a) the location of ① & ②

and not on:

(b) the path from ① to ②

(c) the velocity of the particle

(d) nor on time,

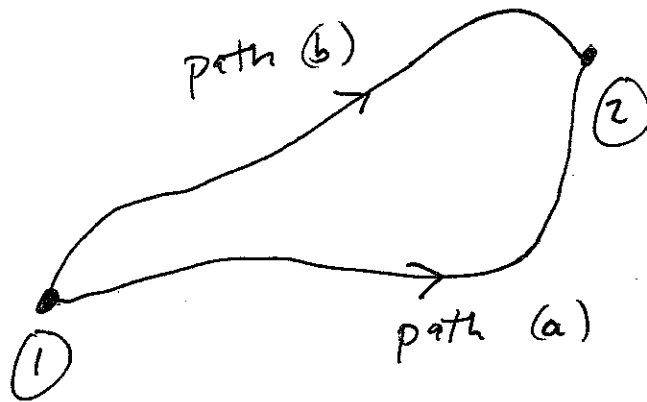
then we say the force is "conservative"

That the ^{work} ~~force~~ should be path-independent allows us to conclude that

$$\vec{\nabla} \times \vec{F} = \text{curl of } \vec{F} = \emptyset$$

via Stokes's Theorem. The proof is as follows:

Assume that ^{the} work between ① & ② is independent of path:



$$\int_{\text{path (a)}}^2 \vec{F} \cdot d\vec{r} \neq \int_{\text{path b}}^2 \vec{F} \cdot d\vec{r} = \emptyset$$

Then
$$W_{12} = \int_{\text{path a}}^2 \vec{F} \cdot d\vec{r} = \int_{\text{path b}}^2 \vec{F} \cdot d\vec{r}$$

If we reverse the direction of path (b), then $d\vec{r}$ becomes $-d\vec{r}$, but \vec{F} stays the same. So

$$\int_{\text{path b}}^2 \vec{F} \cdot d\vec{r} = - \int_{\text{reversed path b}}^1 \vec{F} \cdot d\vec{r}$$

Therefore the total work around path (a) and returning to 1 along reversed path (b) is zero:

$$\int_{\text{path a}}^2 \vec{F} \cdot d\vec{r} - \int_{\text{path b}}^2 \vec{F} \cdot d\vec{r} = \emptyset$$

$$\int_{\text{path a}}^2 \vec{F} \cdot d\vec{r} + \int_{\text{reversed path b}}^1 \vec{F} \cdot d\vec{r} = \emptyset$$

or

$$\oint \vec{F} \cdot d\vec{r} = 0$$

closed path

entire circuit

By Stokes Theorem: $\oint \vec{F} \cdot d\vec{r} = \int_{\text{surface}} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA$

Therefore $\int_{\text{surface}} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA = 0$

Since this must hold true for any path,
it must be true that

$$\boxed{\vec{\nabla} \times \vec{F} = 0} \quad \text{For conservative forces.}$$

Since conservative forces perform work independent of the path (by definition), we can define a function which tells us the work performed starting from any reference point and travelling to any other point. This is the potential energy function.

For convenience we define the potential energy with a minus sign:

$$\Delta T = \int_1^2 \vec{F} \cdot d\vec{r} = W(\vec{r}_1 \rightarrow \vec{r}_2) = -\Delta U$$

or

$$\Delta T + \Delta U = \phi$$

$$\Delta(T+U) = \phi$$

or the mechanical energy is conserved:

$$E \equiv \text{Mechanical energy} \equiv T+U$$

$$\Delta E = \phi$$

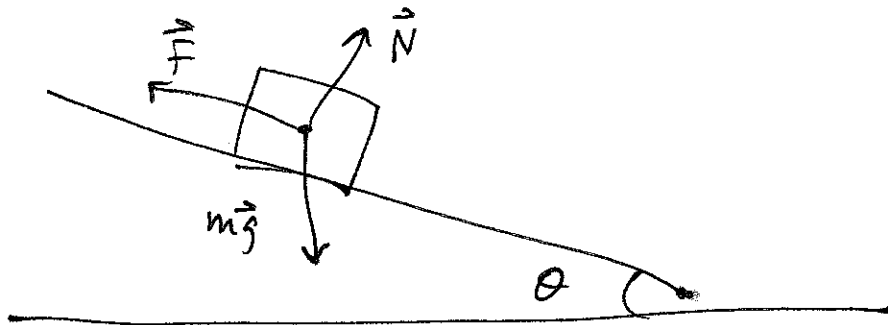
Non-conservative forces

Even if we have a non-conservative force, the work-energy theorem is still useful, because the change in E must be due to the work done by the non-conservative forces.

$$\begin{aligned} \Delta T = W &= W_{\text{conservative}} + W_{\text{non-conservative}} \\ &= -\Delta U + W_{nc} \end{aligned}$$

$$\Delta(T+U) = W_{\text{non-conservative}}$$

Ex: Block sliding down an incline with friction



For a uniform gravitational field, $\vec{F}_{\text{gravity}} = -mg \hat{z}$,

$$\text{So } U_{\text{gravity}} = - \int_{z_0}^{z} (-mg) \hat{z} \cdot \underbrace{\hat{z} dz}_{d\vec{r}}$$

$$U(z) = mg(z - z_0)$$

⇒

Starting from rest and at height z (measured from the bottom of the ramp)

$$\left. \begin{array}{l} U_{\text{initial}} = mgz \\ T_{\text{initial}} = 0 \end{array} \right\} E_i = mgz$$

$$\left. \begin{array}{l} U_{\text{final}} = 0 \\ T_{\text{final}} = \frac{1}{2} m v_f^2 \end{array} \right\} E_f = \frac{1}{2} m v_f^2$$

The change in mechanical energy must be due to friction:

$$\Delta E = \frac{1}{2} m v_F^2 - m g z = W_{\text{fric}}$$

The frictional force is

$$|\vec{F}| = \mu |\vec{N}| = \mu m g \cos \theta$$

It's work is

$$W_{\text{fric}} = (-\mu m g \cos \theta) (d)$$

↑
distance travelled
along the ramp.

The height (z) is related to (d) by geometry:

$$z = d \sin \theta$$

Therefore

$$\frac{1}{2} m v_F^2 - m g (d \sin \theta) = -\mu m g \cos \theta d$$

$$v_F = \sqrt{2 g d (\sin \theta - \mu \cos \theta)}$$

This is the final velocity.

Answers