Kepler's Third Law: $\tau^2 = \dfrac{4\pi^2 \mu a^3}{\gamma}$ ← semi-major axis length

↑ τ = period

Relationship between energy and ε and l:

$$E = \frac{\gamma^2 \mu}{2l^2}\left(\varepsilon^2 - 1\right)$$

Angular change: $$\phi(r) = \int_{r_1}^{r_2} \frac{\frac{l}{r^2}\,dr}{2\mu\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}$$

Angular Momentum and Rigid Bodies

For a fixed rotation axis, and ignoring the components of $\vec{L}$ that are $\perp$ to the rotation axis, we have

$$L_z = I_z \omega \quad , \quad I_z = \int (x^2 + y^2)\, dm = \int r^2\, dm$$
$$= \sum_i m_i r_i^2$$

Then $T = \frac{1}{2} I_z \omega^2$

Including the motion of the CM,

$$\vec{L} = \vec{R}_{cm} \times \vec{P} + \left(I_z^{cm} \omega'\right)\hat{z}$$

↑ rotational angular velocity about the center of mass.

If the rotation axis is not fixed, and/or we desire to know about all 3 components of $\vec{L}$, then we have

$$\vec{L} = I\vec{\omega}\,, \quad \vec{\omega} = \text{angular velocity vector}$$
$$I = \text{inertia tensor}$$

$$I = \begin{pmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{pmatrix}$$

Kinetic Energy: $T = \frac{1}{2}\vec{\omega}\, I\, \vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

Principal Axes: We can always find at least 3 orthogonal directions in space (unit vectors) for which

$$I\hat{e}_1 = \lambda_1 \hat{e}_1 \qquad \hat{e}_1, \hat{e}_2, \hat{e}_3 \text{ are eigenvectors}$$
$$I\hat{e}_2 = \lambda_2 \hat{e}_2 \qquad \lambda_1, \lambda_2, \lambda_3 \text{ are the eigenvalues}$$
$$I\hat{e}_3 = \lambda_3 \hat{e}_3$$

We call the eigenvectors the principal axes, and the $\{\lambda\}$, the principal moments.

If we choose the principal axes as the coordinate system, then I will appear as a diagonal matrix:

$$I = \begin{pmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{pmatrix}$$

In the system of the principal axes,

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$\omega_1, \omega_2, \omega_3$ are the projections of $\vec{\omega}$ onto the principal axes at each moment.

If we project $\frac{d\vec{L}}{dt}$ and $\vec{\omega}$ onto the principal axes at each moment (even though the principal axes keep rotating), we have

$$\left(\frac{d\vec{L}}{dt}\right)_1 = \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2$$

$$\left(\frac{d\vec{L}}{dt}\right)_2 = \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3$$

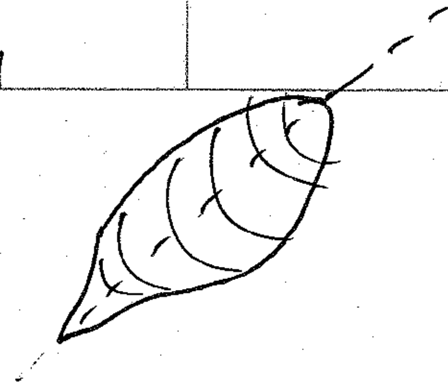
$$\left(\frac{d\vec{L}}{dt}\right)_3 = \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1$$

And since $\vec{\Gamma} = \frac{d\vec{L}}{dt}$ we have

$$\left.\begin{aligned} \Gamma_1 &= \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2 \\ \Gamma_2 &= \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3 \\ \Gamma_3 &= \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1 \end{aligned}\right\} \text{Euler's Equations}$$

These are most useful for the free precession of a top, when $\Gamma_1 = \Gamma_2 = \Gamma_3 = \emptyset$.

Symmetric Free Top:

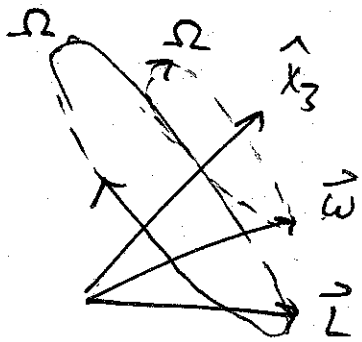


$$I = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Then $\vec{\omega} = \underbrace{(\omega_0 \cos(\Omega t), -\omega_0 \sin(\Omega t), \omega_3)}$

projected onto the rotating body axes

where $\Omega = \left(\frac{I - I_3}{I}\right)\omega_3$



$\hat{x}_3$, $\vec{\omega}$, and $\vec{L}$ remain coplanar. $\vec{\omega}$ & $\vec{L}$ precess about $\hat{x}_3$ as viewed from the body frame

Final Exam Review

Accelerating Frames of Reference:

$$m\ddot{\vec{r}} = \vec{F} \underbrace{- m\vec{A}}_{\text{"pseudo-force"}} , \quad \vec{A} \text{ is acceleration of the frame of reference w/respect to an inertial system}$$

Rotating Frames, described by $\vec{\Omega}$ vector:

Two pseudoforces:

$$\vec{F}_{coriolis} \equiv 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\vec{F}_{centrifugal} \equiv m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

Hamiltonian Mechanics

$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, Then $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$, $i = 1, 2, 3, \ldots$ for each generalized coordinate.

Equations of Motion:

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned}}$$

Conservation of Energy: The Hamiltonian is constant if the Lagrangian has no explicit time dependence. Also, the value of the Hamiltonian is exactly equal