Final Exam Review

Accelerating Frames of Reference:

$$m\ddot{\vec{r}} = \vec{F} - \underbrace{m\vec{A}}_{\text{"pseudo-force"}} \quad , \quad \vec{A} \text{ is acceleration of the frame of reference w/respect to an inertial system}$$

Rotating Frames, described by $\vec{\Omega}$ vector:

Two pseudoforces:

$$\vec{F}_{coriolis} \equiv 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\vec{F}_{centrifugal} \equiv m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

Hamiltonian Mechanics

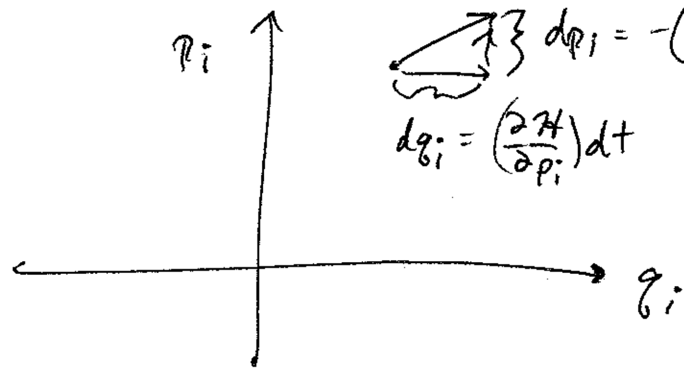
$p_i \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$, Then $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$, $i = 1, 2, 3, \ldots$ for each generalized coordinate.

Equations of Motion:

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned}}$$

Conservation of Energy: The Hamiltonian is constant if the Lagrangian has no explicit time dependence. Also, the value of the Hamiltonian is exactly equal to the energy if the potential is velocity independent and the equations connecting the q_i and (x, y, z) do not depend on time.

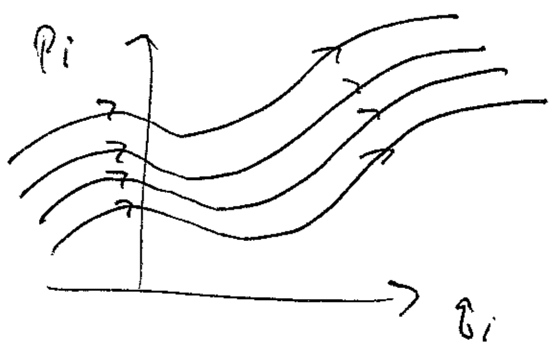
Phase Space:



$$dp_i = -\left(\frac{\partial \mathcal{H}}{\partial q_i}\right) dt$$

$$dq_i = \left(\frac{\partial \mathcal{H}}{\partial p_i}\right) dt$$

There can be no crossings in phase space.



Phase space density is conserved according to Liouville's Theorem

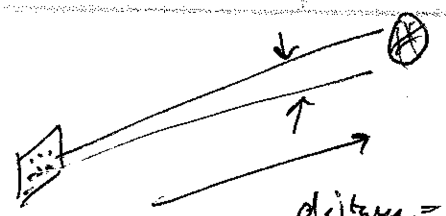
Rutherford Scattering

$$N_{scattered} = (N_{incoming})(n_{target})\,\sigma$$

n_{target}: # of target per unit volume

σ: cross-section, units = m^2 or cm^2 or barn ($10^{-24}\,cm^2$)

Solid Angle:



$$\Delta\Omega = \frac{A}{R^2}$$

Limit when $A \to 0$, then $\Delta\Omega \to d\Omega = \frac{dA}{r^2}$

$$= \sin\theta \, d\theta \, d\phi$$

$$d\sigma(\text{scattering into } d\Omega) = \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

$\frac{d\sigma}{d\Omega}$ = differential cross section, $\sigma_{total} = \int \frac{d\sigma}{d\Omega} \, d\Omega$

$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \frac{d\sigma}{d\Omega}(\theta, \phi)$$

If b is the impact parameter which causes scattering angle θ, then

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

For Rutherford scattering (inverse square law),

$$b(\theta) = \frac{\gamma}{mv^2} \cot\left(\frac{\theta}{2}\right), \quad \gamma = kqQ$$

Then $$\frac{d\sigma}{d\Omega} = \left(\frac{kqQ}{4E\sin^2(\theta/2)} \right)^2$$

an integral of the form

$$\int_{x_1}^{x_2} F\left[y(x), y'(x), x\right] dx$$

is solved by the Euler Lagrange equation

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad \text{where } y' = \frac{dy}{dx}$$

For example, the length of a curve $y(x)$ in the xy plane is given by

$$\int_{x_1}^{x_2} \sqrt{1+\left(y'(x)\right)^2}\, dx$$

To find the minimum length between x_1 & x_2 we solved $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ for $F = \sqrt{1+(y')^2}$.

<u>Central Force Motion</u>

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \text{let } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

$$T = \tfrac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right), \quad \mu = \frac{m_1 m_2}{m_1+m_2} = \text{"reduced mass"}$$

$$\mathcal{L} = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{CM}} + \underbrace{\left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{\mathcal{L}_{relative}}$$

Use Polar Coordinates for the relative motion, with the origin at the CM:

$$\mathcal{L}_{rel} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

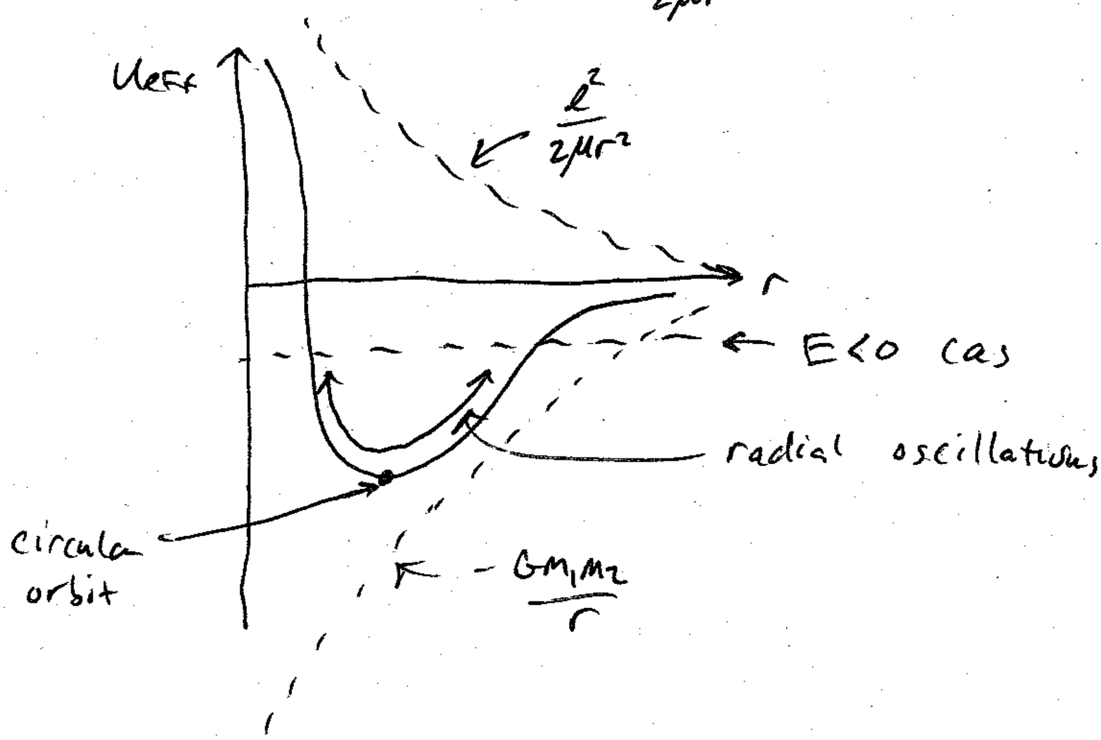
$$\Rightarrow \dot{\phi} = \frac{\ell}{\mu r^2}, \quad \ell = \text{constant} \quad \text{(angular momentum)}$$

Radial Equation:

$$\mu\ddot{r} = -\frac{d}{dr}\underbrace{\left[U(r) + \frac{\ell^2}{2\mu r^2}\right]}_{U_{eff}(r)}$$

Energy conservation: $\tfrac{1}{2}\mu\dot{r}^2 + U_{eff} = \text{constant} = E$.

For gravity, $U_{eff} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$



Orbital Equation: (determining the shape of the orbit)

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u^2(\phi)} F \qquad \text{where } u \equiv \frac{1}{r}$$

For the case of the inverse square law,

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2 \quad , \gamma = G m_1 m_2$$

Then the orbital equation is solved by

$$r(\phi) = \frac{c}{1+\epsilon\cos\phi}$$

where $c = \dfrac{\ell^2}{\gamma\mu}$ = units of length

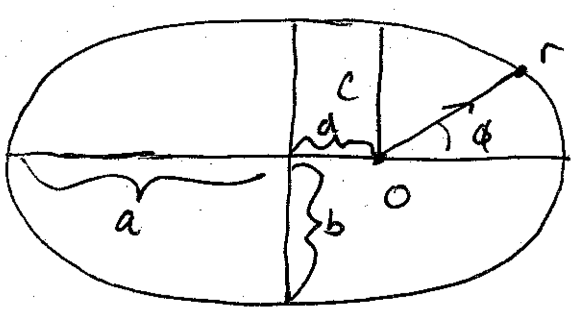
ϵ = eccentricity : $\epsilon = \phi$ (circular orbit)

$\phi < \epsilon < 1$ (elliptical)

$\epsilon = 1$ (parabolic)

$\epsilon > 1$ (hyperbolic)

For an elliptical orbit,



$$a = \frac{c}{1-\epsilon^2}$$

$$b = \frac{c}{\sqrt{1-\epsilon^2}}$$

$$d = a\epsilon$$

Also, $\epsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$

Semi-major axis length

Kepler's Third Law: $\tau^2 = \dfrac{4\pi^2 \mu a^3}{\gamma}$

τ = period

Relationship between energy and ε and ℓ:

$$E = \frac{\gamma^2 \mu}{2\ell^2}\left(\varepsilon^2 - 1\right)$$

Angular change: $$\phi(r) = \int_{r_1}^{r_2} \frac{\frac{\ell}{r^2}\,dr}{2\mu\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}$$

Angular Momentum and Rigid Bodies

For a fixed rotation axis, and ignoring the components of $\vec{L}$ that are $\perp$ to the rotation axis, we have

$$L_z = I_z \omega \quad , \quad I_z = \int (x^2 + y^2)\,dm = \int r^2\,dm$$
$$= \sum_i m_i r_i^2$$

Then $T = \frac{1}{2} I_z \omega^2$

Including the motion of the CM,

$$\vec{L} = \vec{R}_{cm} \times \vec{P} + \left(I_z^{cm} \omega'\right)\hat{z}$$

rotational angular velocity about the center of mass.

Special Relativity

time dilation: $\Delta t = \gamma \Delta t'$, $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}, \ \beta = \frac{v}{c}, \ \eta = \gamma\beta}$

$\Delta t'$ = proper time,
time observed
between 2 events
in a frame where both events occur
at the same spatial location.

Length contraction: $l = \frac{l'}{\gamma}$, l' = proper length
= length observed
in a frame where
The object is at rest.

Lorentz Transformation:

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}, \quad \begin{matrix} \Delta y = \Delta y' \\ \Delta z = \Delta z' \end{matrix}$$

and

$$\begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix}$$

Space time interval: $(\Delta s)^2 \equiv (\Delta x)^2 - (c\Delta t)^2$
= invariant under
Lorentz Transformation.

$(\Delta s)^2 > \emptyset \Rightarrow$ "space-like interval"
$\Rightarrow$ 2 events are causally disconnected. observers may disagree about the ordering of the events

$(\Delta s)^2 = \emptyset \longrightarrow$ "light-like interval"

$(\Delta s)^2 < \emptyset \Rightarrow$ "time-like interval"
$\Rightarrow$ events can be causally related. All observers must agree on the ordering.

Formalism

We look for 4-vectors which transform according to the Lorentz Transformation: $A' = \Lambda A$,
$A =$ 4-vector.

An example is $(x, y, z, ct) \doteq X$
To calculate the "length" we do

$$(\Delta s)^2 = x^2 + y^2 + z^2 - (ct)^2 = X \cdot X$$

Another 4-vector is

$$p = (\gamma m\vec{v}, \gamma mc) = \text{energy momentum 4-vector}$$
$$= (\vec{p}, E/c)$$

Its "length" is called the invariant mass:

$$p \cdot p = \vec{p}^2 - (E/c)^2 = -m^2$$

This is usually written as

$$E^2 = (pc)^2 + (mc^2)^2,$$

or, better yet, set $c = 1$. Then $E^2 = p^2 + m^2$

Also $\beta = \frac{pc}{E}$ or $\beta = \frac{p}{E}$

Also, but less useful, $\vec{p} = \gamma m \vec{v}$ and $E = \gamma m c^2$

Other 4-vectors:

4-velocity: $u \equiv \left(\gamma \frac{d\vec{x}}{dt}, \gamma c\right)$

4-wavenumber: $k \equiv (\vec{k}, \omega/c)$

Velocity transformation: $v_x' = \frac{v_x - V}{1 - v_x V/c^2}$

$$v_y' = \frac{v_y}{\gamma(1 - v_x V/c^2)}$$