Small oscillations

Near a local minimum in the potential,

$$U(x_0+a) \approx U(x_0) + U'(x_0)a + \tfrac{1}{2}U''(x_0)a^2 + \ldots$$

Then $\omega_0 \approx \sqrt{\frac{U''(x_0)}{m}}$

Damped Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \Rightarrow x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

For $\beta < \omega_0$, define $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, so

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

initial conditions

Driving Forces

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \equiv f(t)$$

Then soln is

$$x(t) = \underbrace{x_{\text{particular}}(t)}_{\text{no free parameters}} + \underbrace{x_{\text{transient}}(t)}_{\text{2 free parameters}}$$

For $F(t) = F_0 \cos(\omega t)$,

$$X_{particular}(t) = A\cos(\omega t - \delta),$$

$$A = \frac{F_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$

Periodic Driving Forces

Let $F(t) = \sum_{n=0}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$

where $\omega = \frac{2\pi}{T}$ and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos(n\omega t)\,dt \quad , n \geq 1$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega t)\,dt \quad , n \geq 1$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} F(t)\,dt$$

If $F(t)$ has cosine terms only, then

$$X(T) = \sum_n X_n(t) \quad , X_n(t) = A_n \cos(n\omega t - \delta_n),$$

$$A_n = \frac{a_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}}$$

$$\text{and} \quad \delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$

Green's Method for a damped oscillator, arbitrary forcing function.

For a forced, damped oscillator,

$$x(t) = \int_{-\infty}^{t} F(t')\, G(t,t')\, dt' \, ,$$ where

$$G(t,t') = \begin{cases} \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t')) , & \text{for } t \geq t' \\ \emptyset & , \text{for } t < t' \end{cases}$$

Lagrangian Mechanics

$\mathcal{L} = T - U$. Then

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

If q does not appear in the Lagrangian, then we say that q is "ignorable" or "cyclic". This leads to a conservation law.

Exam Review

~~[illegible]~~

Hamilton's principle

The "action" is $S \equiv \int_{t_1}^{t_2} \mathcal{L}(\dot{x}, x, t)\, dt$, $\mathcal{L}$ = Lagrangian

We examine small variations around the true path

$$X(t) = \underset{\substack{\uparrow \\ \text{true path}}}{x(t)} + \underset{\substack{\uparrow \\ \text{small parameter}}}{\alpha}\, \eta(t) \leftarrow \text{variation}$$

We say the action is stationary if S has no first-order dependence on the small parameter α. Hamilton's Principle says that the action should be stationary: $\delta S = 0$

This leads to the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \quad \text{for } x$$

or $$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad \text{for a generalized coordinate } q_i.$$

Calculus of Variations

Any mathematical problem which can be formulated as ~~the~~ finding a maximum or minimum value for an integral of the form

$$\int_{x_1}^{x_2} F\left[y(x), y'(x), x\right] dx$$

is solved by the Euler Lagrange equation

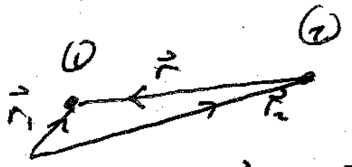
$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad \text{where} \quad y' = \frac{dy}{dx}$$

For example, the length of a curve $y(x)$ in the xy plane is given by

$$\int_{x_1}^{x_2} \sqrt{1 + \left(y'(x)\right)^2}\, dx$$

To find the minimum length between x_1 & x_2 we solved $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ for $F = \sqrt{1+(y')^2}$.

Central Force Motion



$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \text{let } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

$$T = \tfrac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right), \quad \mu = \frac{m_1 m_2}{m_1+m_2} = \text{"reduced mass"}$$

$$\mathcal{L} = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{cm}} + \underbrace{\left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 + U(r)\right)}_{\mathcal{L}_{relative}}$$