Relativistic velocity Addition

In frame S, $\vec{v} = \dfrac{d\vec{r}}{dt}$

In frame S',
$$dx' = \gamma(dx - V\,dt)$$
$$dy' = dy$$
$$dz' = dz$$
$$dt' = \gamma(dt - V\,dx/c^2)$$

so
$$v_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - V\,dt)}{\gamma(dt - V\,dx/c^2)}$$

or
$$\boxed{v_x' = \frac{v_x - V}{1 - v_x V/c^2}}$$

Also
$$v_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - V\,dx/c^2)}$$

$$\boxed{v_y' = \frac{v_y}{\gamma(1 - v_x V/c^2)}}, \quad \boxed{v_z' = \frac{v_z}{\gamma(1 - v_x V/c^2)}}$$

where $\gamma = \dfrac{1}{\sqrt{1 - V^2/c^2}}$

Example: A bullet is fired at speed $0.6c$ from a rocket traveling at $0.8c$. Relative to the earth, how fast does the bullet travel?

Answer: Our usual convention is that S is fixed and S' is moving. We can infer that

$$v = \frac{v' + V}{1 + v'V/c^2}$$

by reversing the sign of V and exchanging $v' \leftrightarrow v$.

Then $$v = \frac{0.6c + 0.8c}{1 + (0.8)(0.6)} = \frac{1.4c}{1.48} = 0.95c.$$

So the velocity is less than c as measured by earth.

4-velocity

The three-velocity is $\vec{v} = \dfrac{d\vec{x}}{dt}$,

where dt is the time as seen by a fixed observer. We can construct a 4-vector for the velocity by considering a related ~~ed~~ quantity

$$\vec{u} \equiv \frac{d\vec{x}}{dt_0},$$

where dt_0 is the proper time. Since the proper time a Lorentz Invariant, the vector $\vec{u}$ is useful for defining a 4-vector for velocity.

We consider the 4-velocity to be

$$u = \frac{dx}{dt_0} = \left(\frac{d\vec{x}}{dt_0}, c\frac{dt}{dt_0}\right)$$

$$\boxed{u = \left(\gamma \frac{d\vec{x}}{dt}, \gamma c\right) = \text{4-velocity}}$$

where $\gamma = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}$

Notice that the ordinary 3-velocity is not simply the first three components of the 4-velocity. Instead the 3-velocity is the first 3 components divided by γ, $\left(\gamma = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}\right)$

The 4-velocity is useful primarily for defining the 4-momentum:

$$\boxed{p \equiv mu = (\gamma m\vec{v}, \gamma mc) = \text{4-momentum}}$$

Since m is a lorentz scalar (Lorentz Invariant), and u is a 4-vector, this means that p is a Lorentz 4-vector.

~~We~~ also

We define the energy such that the 4th component of the 4 momentum is E/c:

$$E/c \equiv p_4 = \gamma mc$$

or $\boxed{E = \gamma mc^2}$

Then the 4-momentum is

$$p = (\vec{p}, E/c)$$

where $\vec{p} = \gamma m \vec{v}$.

Does our definition of E make sense? Let's expand

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \ldots\ldots$$

Then $E = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\ldots\right)(mc^2)$

$$= mc^2 + \frac{1}{2}mv^2 + \ldots\ldots$$

where mc^2 is the rest-mass energy and $\frac{1}{2}mv^2 + \ldots\ldots$ is the kinetic energy.

The important feature of this result is that in some reactions, such as particle decays the rest mass energy can change. This is fine as long as the relativistic energy E is conserved. So kinetic energy can become rest-mass energy and vice-versa.

Apparently we have $E = mc^2 + T$, $T \to$ kinetic energy

$$\text{or} \quad \boxed{T = E - mc^2 = (\gamma - 1)mc^2}$$

The kinetic energy T is only a useful quantity if you know that m has not changed. In this case T must be conserved so that E is conserved.

Summarizing, we have two new 4-vectors.

4-velocity : $u = (\gamma\vec{v}, \gamma c)$

4-momentum : $p = (\vec{p}, E/c)$

where $\vec{p} = \gamma m \vec{v}$.

Since these quantities are 4-vectors, they transform according to the Lorentz Transformation:

$$\begin{pmatrix} \gamma v_x' \\ \gamma c \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \gamma v_x \\ \gamma c \end{pmatrix} \Bigg\} \text{4-velocity transformation}$$

and

$$\begin{pmatrix} p_x' \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ E/c \end{pmatrix} \Bigg\} \text{4-momentum transformation}$$

Invariant mass

Note that the length of the 4-momentum is simply the negative of the invariant mass. We can see this by considering the

4-momentum as it appears in the particle's rest-frame. In this frame,

$$p = (\emptyset, \emptyset, \emptyset, mc).$$

So $\quad p \cdot p = (\emptyset)(\emptyset) - (mc)^2 = -(mc)^2$

But since p is a 4-vector, its length must be the same in all reference frames. So the length is $-(mc)^2$ always.

We can go a step further by noting that in any other frame where the particle is moving we have

$$p = (\vec{p}, E/c)$$

$$p \cdot p = \vec{p}^{\,2} - \left(\frac{E}{c}\right)^2$$

And this must be equal to $-(mc)^2$:

$$\vec{p}^{\,2} - \left(\frac{E}{c}\right)^2 = -(mc)^2$$

or $\boxed{E^2 = (pc)^2 + (mc^2)^2}$ This is the most useful relation in all of relativistic kinematics.

We usually try to avoid thinking about velocity, and instead think about E, p, and m, which are related by this formula.

In those instances where we want to know the velocity, we calculate it like this:

$$\frac{pc}{E} = \frac{(\gamma m v)c}{\gamma m c^2} = \frac{v}{c} = \beta$$

or $$\boxed{\beta = \frac{pc}{E}}$$

This is much better than trying to calculate v or β by solving for β in $\frac{1}{1-\beta^2}$,

because for high energy particles ~~the~~ a typical calculator will not be accurate enough.

Note that if we choose our system of units such that the speed of light $= 1$, then our kinematic formulas are very simple:

$$E^2 = p^2 + m^2, \quad \beta = \frac{p}{E}$$

$$p = (\vec{p}, E), \text{ and } u = (\gamma\vec{\beta}, \gamma)$$

$$\text{or } u = (\vec{\eta}, \gamma)$$

where $\vec{\beta} = \left(\frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right)$

and $\vec{\eta} = \vec{\beta}\gamma$

Summary of Relativistic Kinematics.

1) Always use $\boxed{E^2 = (pc)^2 + (mc^2)^2}$

or $E^2 = p^2 + m^2$ (with $c = 1$)

2) Avoid using $E = \gamma mc^2$

or $E = \gamma m$

unless you already know γ.

3) To get the velocity use

$$\beta = \frac{pc}{E}$$

or $$\beta = \frac{p}{E}$$

4) Avoid getting β from γ.

~~Examples~~

5) Energy-momentum conservation means

$$\underbrace{P_{initial} = P_{Final}}_{\text{4-vectors}}$$

or $$(p_x, p_y, p_z, E/c)_{initial} = (p_x, p_y, p_z, E/c)_{final}$$

6) From energy-momentum conservation, it must be true that

$$\underbrace{|P_{initial}|^2}\ = \underbrace{|P_{Final}|^2}$$

Invariant mass is the same before and after

But the invariant mass is not just the simple sum of the masses of the particles.

Ex: $\pi^+ \rightarrow \mu^+ \nu_\mu$

π^+: $M = 140$ MeV; μ^+: $M = 106$ MeV; ν_μ: $M = 0$

So it's not true that $M_\pi = M_\mu + M_\nu$

To see why, look at the final state 4-momentum:

$$P_{Final} = (0, 0, 0, (E_\mu + E_\nu)/c)$$

$$(|M_{invariant}|c)^2 = -\frac{(E_\mu + E_\nu)^2}{c^2}$$

$$= -\frac{(E_\mu^2 + E_\nu^2 + 2E_\mu E_\nu)}{c^2}$$

$$= -\frac{((m_\mu c^2)^2 + (p_\mu c)^2 + (p_\nu c)^2 + 2E_\mu E_\nu)}{c^2}$$

The μ^+ momentum and ν momentum contribute to the final state invariant mass. It's not just $M_\mu + M_\nu$!!

Compton Scattering

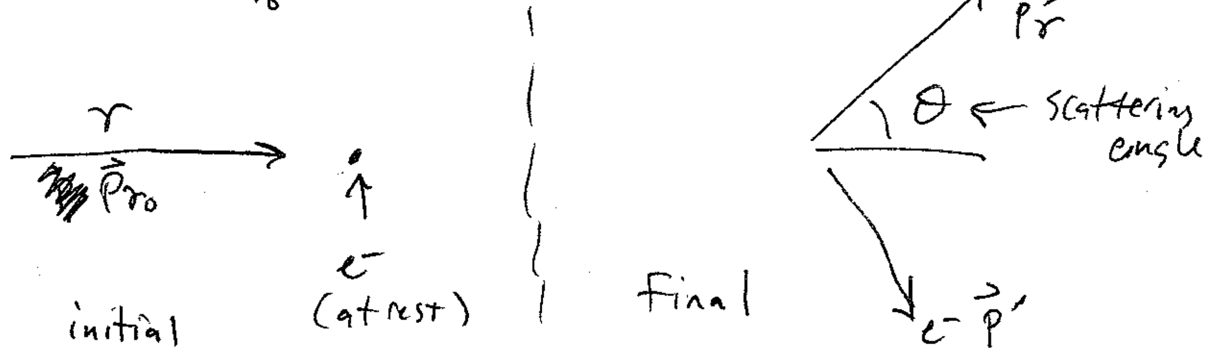
One of the most compelling pieces of evidence for the existence of photons - individual particles of light - comes from the scattering of photons on atomic electrons. The question is: do these photons really behave like small billiard balls, conserving momentum and energy in the collision?

The electron is considered to be at rest in the initial state, so its 4-momentum is

$$p_e = (\emptyset, \emptyset, \emptyset, m_e c) \quad \left(\text{or } p_e = (\emptyset, \emptyset, \emptyset, m_e)\right)$$

In practice the electron has a ~~small~~ non-zero expectation value to have a small amount of momentum while bound to the atom, but we neglect this in comparison with the energy and momentum of the incoming photon (gamma ray). The gamma ray 4-momentum is

$$p_{\gamma 0} = (\vec{p}_{\gamma_0}, E_{\gamma_0}/c)$$



According to the de Broglie hypothesis,

$$\vec{p}_0 = \hbar \vec{k}_0 \quad \text{and} \quad E = \hbar\omega_0, \text{ so}$$

$$p_{\gamma 0} = \left(\hbar \vec{k}_0, \frac{\hbar\omega_0}{c}\right) = \hbar\left(\vec{k}_0, \frac{\omega_0}{c}\right)$$

we can re-write in terms of unit vector $\hat{k}_0$:

$$|\vec{k}_0| = \frac{\omega_0}{c}, \quad \text{so} \quad \vec{k}_0 = \frac{\omega_0}{c}\hat{k}_0$$

Then

$$p_{\gamma 0} = \frac{\hbar\omega_0}{c}(\hat{k}_0, 1) \Leftarrow \text{initial gamma 4-momentum}$$

Then the final γ 4-momentum is

$$p'_\gamma = \frac{\hbar\omega'}{c}(\hat{k}', 1)$$

where $\omega' \neq \omega_0$, because the gamma has lost some energy.

The final electron 4-momentum is

$$p'_e = (\vec{p}', E'_e)$$

So we have

$$\underbrace{p_e + p_{\gamma 0} = p'_e + p'_\gamma}_{\text{conservation of 4-momentum}}$$

or $p'_e = p_e + (p_{\gamma 0} - p'_\gamma)$

Now we square both sides.

$$(p_e')^2 = p_e' \cdot p_e' = \left(p_e + (p_{\gamma 0} - p_\gamma')\right) \cdot \left(p_e + (p_{\gamma 0} - p_\gamma')\right)$$

$$= p_e^2 + 2p_e(p_{\gamma 0} - p_\gamma') + (p_{\gamma 0}^2 - 2p_{\gamma 0} \cdot p_\gamma' + p_\gamma'^2)$$

Now things simplify. We know that

$$\left.\begin{aligned} p_e' \cdot p_e' &= -m_e c \\ \text{and } p_e \cdot p_e &= -m_e c \end{aligned}\right\} \text{rest mass energy of the electron}$$

So these terms cancel.

$$\text{Also, } \left.\begin{aligned} p_{\gamma 0}^2 &= 0 \\ p_\gamma'^2 &= 0 \end{aligned}\right\} \text{rest mass energy of the photon.}$$

So we are left with

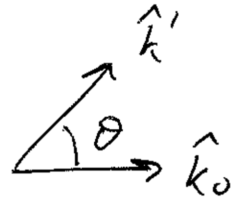
$$0 = 2p_e \cdot (p_{\gamma 0} - p_\gamma') - 2p_{\gamma 0} \cdot p_\gamma'$$

$$\text{or } p_{\gamma 0} \cdot p_\gamma' = \underset{\substack{\uparrow \\ p_e = (0,0,0,m_e c)}}{p_e} \cdot (p_{\gamma 0} - p_\gamma') \qquad \boxed{\frac{\hbar}{c}(\omega_0 \hat{k}_0 - \omega' \hat{k}', \omega_0 - \omega')}$$

$$= -m_e c \left(\frac{\hbar}{c}\right)(\omega_0 - \omega')$$

Left hand side:

$$p_{\gamma_0} \cdot p_{\gamma'} = \frac{\hbar\omega_0}{c}(\hat{k}_0, 1) \cdot \frac{\hbar\omega'}{c}(\hat{k}', 1)$$
$$= \left(\frac{\hbar}{c}\right)^2 \omega_0\omega' \left[\hat{k}_0 \cdot \hat{k}' - 1 \right]$$



$$\hat{k}_0 \cdot \hat{k}' = \cos\theta$$

$$= \left(\frac{\hbar}{c}\right)^2 \omega_0\omega' (\cos\theta - 1)$$

Therefore

$$\left(\frac{\hbar}{c}\right)^2 \omega_0\omega' (\cos\theta - 1) = -m_e c\left(\frac{\hbar}{c}\right)(\omega_0 - \omega')$$

$$\boxed{\frac{\hbar}{m_e c^2}(1 - \cos\theta) = \frac{1}{\omega'} - \frac{1}{\omega_0}}$$

This relates the change in frequency to the scattering angle θ

In terms of wavelength, we have

$$\omega = \frac{2\pi c}{\lambda}$$

or

$$\boxed{\frac{h}{m_e c}(1 - \cos\theta) = \lambda' - \lambda_0}$$

This relates the change in wavelength to the scattering angle. Historically, the experimental confirmation of this formula confirmed the picture of a photon as a particle with $m = \phi$.

Doppler Effect

Any plane wave can be written as

$$\phi = A\cos(\vec{k}\cdot\vec{x} - \omega t)$$

For a light wave, we also have

$$|\vec{k}| = \frac{2\pi}{\lambda}, \quad \text{and} \quad \omega = c|\vec{k}|$$

The phase of the plane wave must be the same in all frames of reference. For example, we can measure the phase by doing an interference experiment. All observers should agree whether the result is constructive interference or destructive interference, otherwise the laws of physics will be inconsistent from one frame to the next

Since $x = (\vec{x}, ct)$ is a 4-vector,

and since $\vec{k}\cdot\vec{x} - \omega t$ should have the same value in all frames of reference, we can define a new 4-vector

$$k \equiv (\vec{k}, \omega/c)$$

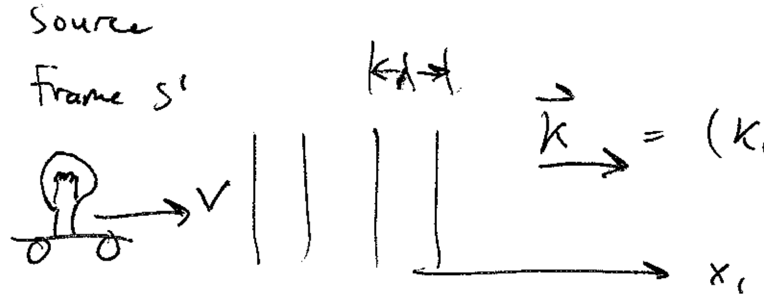
Then $k\cdot x = \vec{k}\cdot\vec{x} - \left(\frac{\omega}{c}\right)(ct) = \vec{k}\cdot\vec{x} - \omega t$

$= $ Lorentz invariant phase.

We can derive the Doppler effect for light from the fact that k is a 4-vector.
According to the Lorentz Transformation,

$$k_4' = \frac{\omega'}{c} = |\vec{k}'| = \gamma(k_4 - \beta k_1)$$
$$= \gamma\left(\frac{\omega}{c} - \beta k_1\right)$$

For V along the direction of $\hat{k}$, we have



$$\vec{k} = (k_1, 0, 0), \quad |\vec{k}| = k_1 = k$$

Then
$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \beta k\right) = \gamma\left(\frac{\omega}{c} - \beta\frac{\omega}{c}\right)$$
$$= \gamma\left(\frac{\omega}{c}\right)(1-\beta)$$

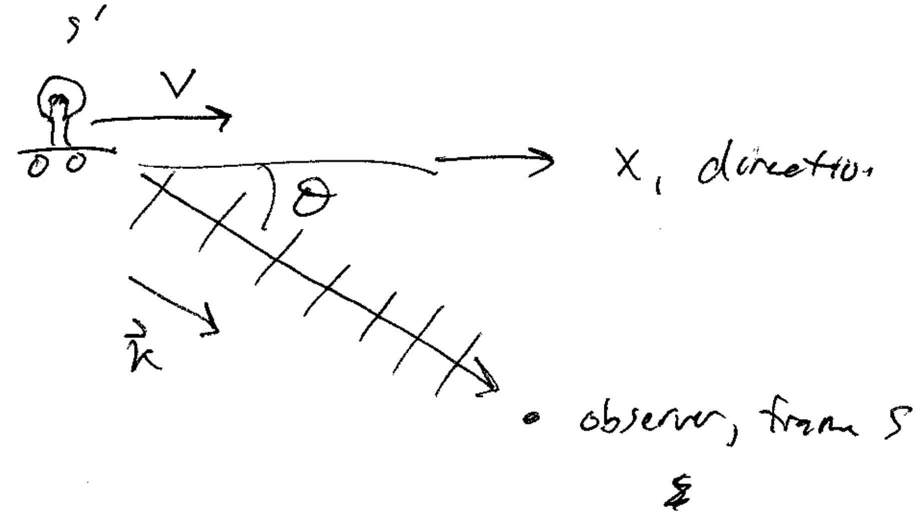
or $\omega' = \gamma\omega\,(1-\beta)$

Let ω_0 be the frequency in frame S' (where the source is at rest). Then $\omega' = \omega_0$.
Solving for ω (frequency in frame S):

$$\boxed{\omega = \frac{\omega_0}{\gamma(1-\beta)}}$$

For ~~travel~~ relative motion in the direction of k.

In general the observer is located at some angle with respect to the motion:



Then $k_1 = |\vec{k}|\cos\theta$, and the result becomes

$$\boxed{\omega = \frac{\omega_0}{\gamma(1-\beta\cos\theta)}}$$

for observer at angle θ to the direction of motion.

Note that for $\theta = \phi$, we have

$$\omega = \frac{\omega_0}{\gamma(1-\beta)} = \frac{\omega_0}{\frac{(1-\beta)}{\sqrt{1-\beta^2}}} = \frac{\omega_0}{\frac{1-\beta}{\sqrt{(1-\beta)(1+\beta)}}} \boxed{= \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_0}$$

For $\theta = \pi/2$, the only effect is time dilation (no length contraction along the direction $\perp$ to the direction of motion.)

Then we have

$$\boxed{\omega = \frac{\omega_0}{\gamma}} \qquad (\theta = \pi/2)$$

This is simply time dilation.