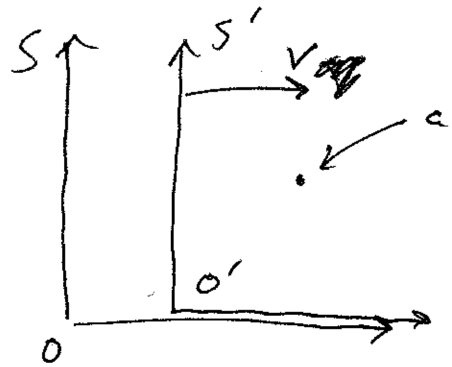
Galilean Relativity:



a point (x) or (x')

$$x' = x - vt \Rightarrow \dot{x}' = \dot{x} - v$$
$$y' = y \qquad \text{and } \ddot{x}' = \ddot{x}$$
$$z' = z$$

$$\Rightarrow \boxed{\begin{aligned} m\vec{a}\,' &= m\vec{a} \\ \vec{F}\,' &= \vec{F} \end{aligned}}$$

Newton's 2nd Law appears to ~~b~~ have the same form in both systems

However, Maxwell's Eqs appear to violate Galilean Relativity. The problem is that a ~~Fundamental~~ velocity appears explicitly → the speed of light. This happens when deriving the wave Eq.

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}, \quad \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}.$$

In which frame does this wave Eq. hold true with $c = 3 \times 10^8$ m/s? Mechanical waves require a medium, the velocity is measured w/respect to the medium. Suppose that such a medium exists for E&M waves, and suppose an observer travels at the speed of light through that medium.

Then the observer sees a "frozen" EM wave. But such a wave violates basic laws of electrostatics such as

$$\vec{\nabla} \times \vec{E} = \emptyset \qquad \left(\text{or } \oint \vec{E} \cdot d\vec{\ell} = \phi\right)$$

because the fixed observer sees

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (\text{Faraday's Law})$$

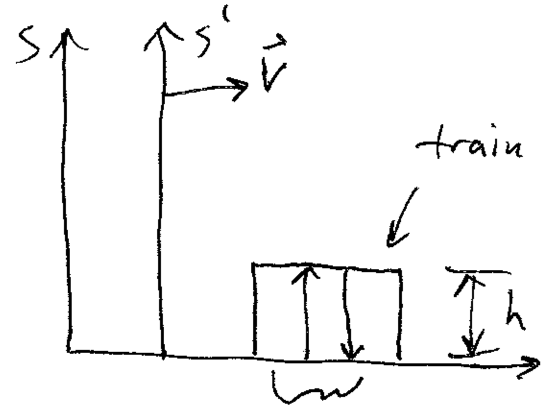
Both observers should observe the $\vec{E}$ field having the same curl ($\vec{\nabla} \times \vec{E}$), because this is just a spatial derivative, which should not change in Galilean Relativity. But the fixed observer sees the magnetic field changing, which satisfies Faradays law, while the moving observer sees nothing changing, so $\frac{\partial \vec{B}}{\partial t} = \emptyset$, violating Faraday's Law.

To fix this, Maxwell's Equations would need to be modified. Einstein and others suggested that we should instead modify our concept of space & time and give up <u>Galilean Relativity</u>. Instead we will modify mechanics to be consistent with the principles of Special Relativity.

1. Every inertial frame is equivalent ⇒ All laws of physics appear to be the same to all inertial observers.

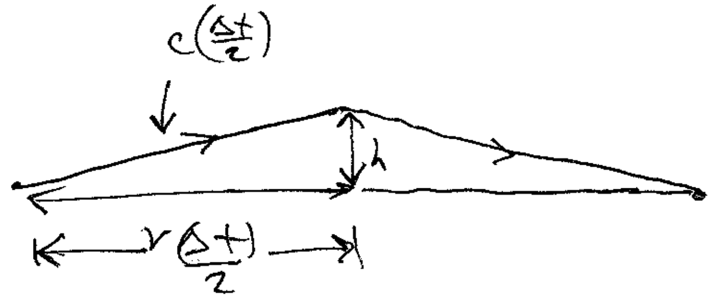
2. The velocity of light is $c = 3\times 10^{8}$ m/s observed in all inertial frames (violating Galilean Relativity where ~~…~~ $\dot{x}' = \dot{x} \pm v$.)

Time Dilation



In S': $\Delta t' = \frac{2h}{c}$

In S:



$$\left(c\left(\frac{\Delta t}{2}\right)\right)^2 = \left(v\frac{(\Delta t)}{2}\right)^2 + h^2$$

or $\Delta t = \dfrac{2h}{\sqrt{c^2-v^2}} = \underbrace{\dfrac{2h}{c}}_{\Delta t'}\dfrac{1}{\sqrt{1-\beta^2}}$, $\boxed{\beta \equiv \dfrac{v}{c}}$

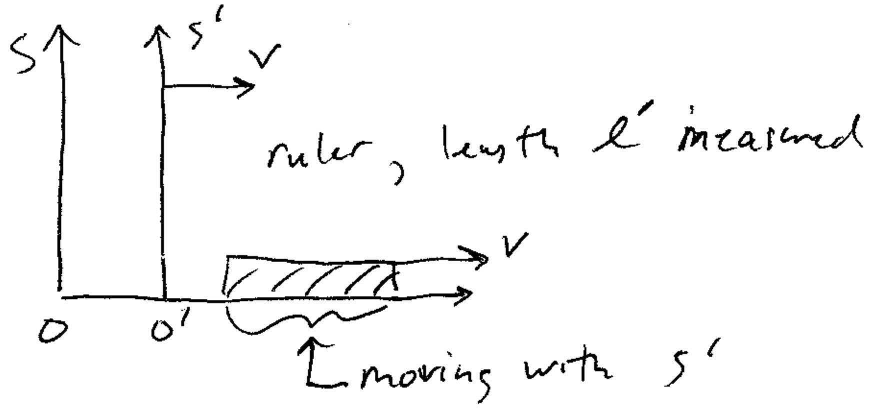
or $\boxed{\Delta t = \dfrac{\Delta t'}{\sqrt{1-\beta^2}}} \equiv \boxed{\gamma(\Delta t'),}$ where $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}}$

So the two observers measure different times for the light to travel up and back.
Two events observed at the same location in space have ~~in the rest frame (where the~~ light time difference ($\Delta t'$).

we call this the proper time.

Length Contraction

A ruler is moving in frame S, but at rest in S':



How long is the ruler as seen in S?
How long does it take the ruler to pass by a location fixed in S (such as the origin)?
We can multiply that time by the velocity v to get the ruler's length in S.

In S', an observer sees the ruler at rest with length ℓ', but also sees a point fixed in S as traveling with speed $|v|$.

$$\ell' = |v|(\underbrace{\Delta t'})$$

the time that a point fixed in S takes to pass the full length of the ruler, as observed in S'.

In S, an observer watches the ruler pass by a fixed location, taking time Δt:

$$\ell = |v|(\Delta t)$$

length in S

time for ruler to pass by in S.

Since the observer in S sees the two events at the same location in space. So Δt is the proper time in this case. Thus

$$\Delta t' = \gamma(\Delta t), \quad \text{so} \quad \ell' = |v|\,\gamma(\Delta t)$$

$$\ell' = \gamma \underbrace{|v|(\Delta t)}_{\ell}$$

$$\ell' = \gamma \ell$$

or $$\boxed{l = \frac{l'}{\gamma} \leq l'}$$

The frame S' is special in this experiment, because the ruler is at rest there we call the length of the ruler observed at rest the "proper length", and use symbol l_0 so

$$\boxed{l = \frac{l_0}{\gamma} \leq l_0}$$

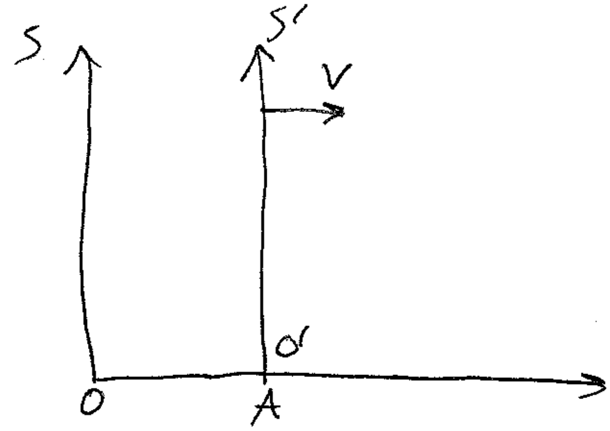
The length observed when the ruler is in motion is shorter by factor $\frac{1}{\gamma}$

(note that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is ≥ 1).

Lorentz Transformation

Let Δx and Δt be the spatial difference and time difference between 2 events measured in S. (And let $\Delta x'$ and $\Delta t'$ be the quantities measured in S'.) We wish to determine a transformation matrix

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$



Event 1: O' and O' coincide

Event 2: O' and A coincide

proper time

In S: $\Delta x = V\Delta t$

In S': $\Delta x' = \emptyset$, $\Delta t' = (\Delta t)\sqrt{1-\beta^2} = \dfrac{\Delta t}{\gamma}$

$\uparrow$

both happen at O', which is fixed in S'

Our tranformation reads as

$$\Delta x' = a_1 \Delta x + a_2 (c\Delta t)$$

$$\Downarrow$$

$$\emptyset = a_1 \Delta x + a_2 (c\Delta t)$$

$$\Rightarrow \quad \frac{a_2}{a_1} = -\frac{\Delta x}{c\Delta t} = -\frac{V}{c} = -\beta$$

For the transformation of $c\Delta t'$, we have

$$c(\Delta t') = a_3 \Delta x + a_4 c(\Delta t)$$

$$c(\Delta t') = a_3 (V\Delta t) + a_4 (c\Delta t)$$

$$c\underbrace{(\Delta t')}_{(\Delta t)\sqrt{1-\beta^2}} = (a_3 V + a_4 c)\Delta t$$

$$c\sqrt{1-\beta^2} = a_3 v + a_4 c$$

$$\sqrt{1-\beta^2} = a_3\beta + a_4$$

So our transformation appears as

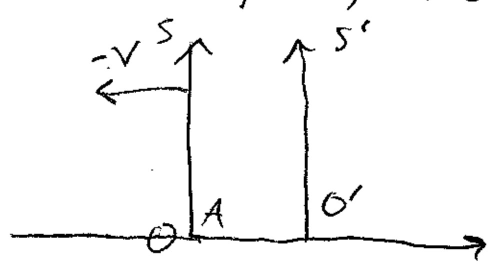
$$\begin{pmatrix} \emptyset \\ \sqrt{1-\beta^2} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Now reverse our point of view. We now see S' as fixed, and S travels backwards at speed $-v$ along $-\hat{x}$. Then we can repeat the above scenario, with everything the same except

$\Delta x \leftrightarrow \Delta x'$

$\Delta t \leftrightarrow \Delta t'$

$v \leftrightarrow -v$.



Then in S', $\Delta x' = -v\Delta t'$

in S, $\Delta x = \emptyset$, while $\Delta t = \Delta t'\sqrt{1-\beta^2}$

proper time

The transformation appears as

$$\begin{pmatrix} -v\Delta t' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ c\Delta t'\sqrt{1-\beta^2} \end{pmatrix}$$

Divide everything by $c\Delta t'$.

$$\begin{pmatrix} -\beta \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{1-\beta^2} \end{pmatrix}$$

Now we can read off the following:

$$-\beta = a_2 \sqrt{1-\beta^2} \implies \boxed{a_2 = \frac{-\beta}{\sqrt{1-\beta^2}}}$$

And we already know $\frac{a_2}{a_1} = -\beta$, so we must have

$$\boxed{a_1 = \frac{1}{\sqrt{1-\beta^2}}}$$

Similarly we can read off

$$1 = a_4 \sqrt{1-\beta^2} \implies \boxed{a_4 = \frac{1}{\sqrt{1-\beta^2}}}$$

From before we have

$$\sqrt{1-\beta^2} = a_3\beta + a_4 = a_3\beta + \frac{1}{\sqrt{1-\beta^2}}$$

This is solved by $\boxed{a_3 = \frac{-\beta}{\sqrt{1-\beta^2}}}$

Finally the Lorentz Transformation is

$$\boxed{\begin{pmatrix} \Delta x' \\ c\,\Delta t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ c\,\Delta t \end{pmatrix}}$$

or

$$\boxed{\begin{aligned} \Delta x' &= \gamma(\Delta x - v\,\Delta t) = \frac{\Delta x - v\,\Delta t}{\sqrt{1-\beta^2}} \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1-\beta^2}} \end{aligned}}$$

The inverse transformation replaces primed variables by unprimed variables and v by $-v$:

$$\boxed{\begin{pmatrix} \Delta x \\ c\,\Delta t \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x' \\ c\,\Delta t' \end{pmatrix}}$$

Sometimes we use the notation $\eta \equiv \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}$

Then the transformation matrix is

$$\begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix}$$

Also notice that $\gamma^2 - \eta^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = 1$

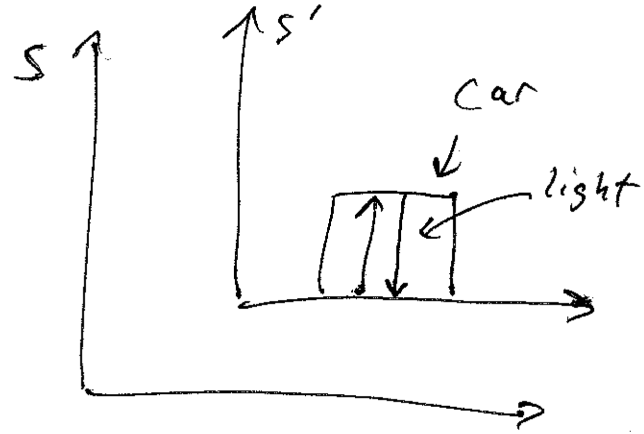
$$\boxed{\gamma^2 - \eta^2 = 1}$$

Now that we have the Lorentz Transformation it is easy to re-derive time dilation and length contraction:

Time Dilation from the Lorentz Transformation

In S',
$\Delta x'$ between light beam leaving and returning is zero: $\Delta x' = \emptyset$

$\Delta t'$ = nonzero.



What is Δt in S? Use the Lorentz Transformation

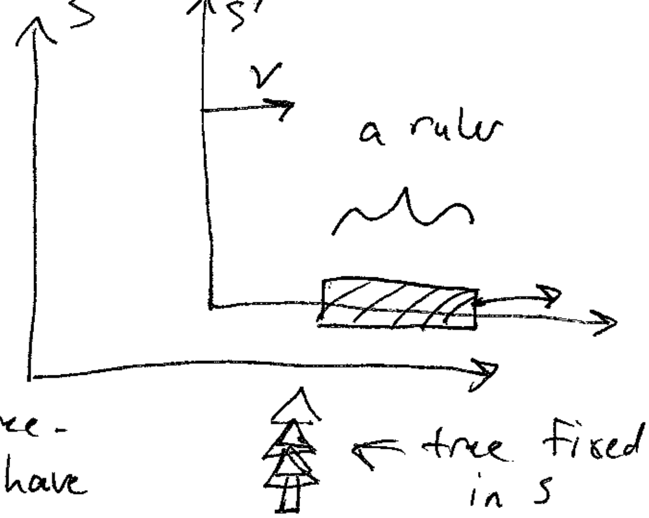
$$\begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x' = \emptyset \\ c\Delta t' \end{pmatrix} \Rightarrow \boxed{\Delta t = \gamma \Delta t'} > \Delta t'$$

proper time

dilated time

Length Contraction from Lorentz Transformation.

In S, the length is $\ell = v(\Delta t)$, where Δt is time between front and back of the ruler being next to the fixed tree. These two spacetime events have $\Delta x = \emptyset$.



In S', the length is ℓ', which is the proper length (because the ruler is fixed in S').

We can calculate ℓ' in terms of the Δx and Δt:

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \emptyset \\ c\Delta t \end{pmatrix}$$

$$\Rightarrow \Delta x' = \text{proper length} \equiv \ell' = -\eta c\Delta t$$

$$= -\underbrace{(\beta c)}_{v}\gamma\Delta t$$

$$= -\underbrace{(v)(\Delta t)}_{\ell}\gamma$$

$$= -\ell\gamma$$

$$\Rightarrow \boxed{\text{proper length} \left|\frac{(\ell')}{\ell}\right| = \gamma} > 1$$

$$\Rightarrow \ell < \ell'$$

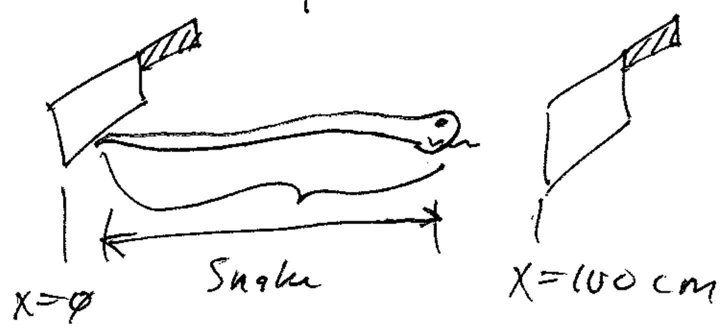
Simultaneous Events

Because of time dilation, observers in different frame cannot all agree on what events occur simultaneously. ⇒ time is relative (depends on your frame of reference.)

Snake paradox (conundrum)

A lab has 2 cleavers set 100 cm apart.
~~A 140 cm snake~~
A snake whose proper length is 100 cm travels between the cleavers at velocity $v = 0.6\,c$ ($\beta = 0.6$). ~~The~~ In the lab frame the cleavers come down when the snake's tail first clears the left cleaver:

Lab Frame point of view:

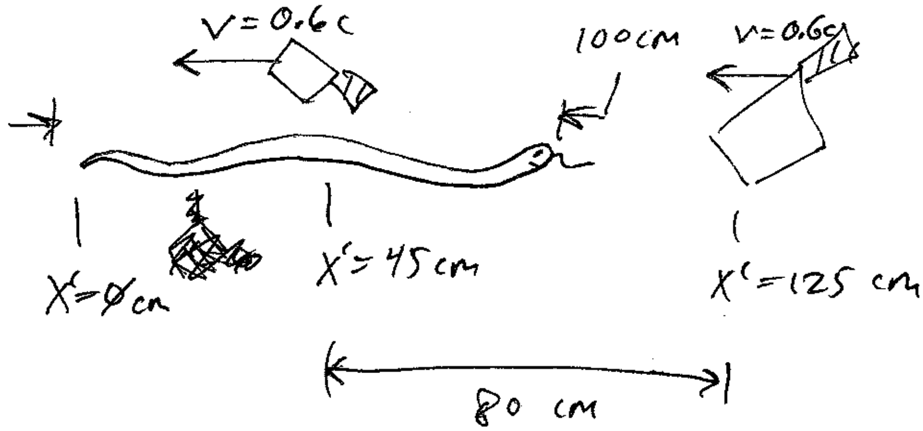


Snake appears 80 cm long in lab frame, so it is safe.

~~To~~ However, isn't it true that in the snake's frame, the cleavers appear only 80 cm apart, while the snake is 100 cm long? Answer: yes, this is how the snake see the situation. So does the snake get cut from its point of view?

Answer: No. From the snake's point of view, the two cleavers do not come down at the same time:

At $t' = -2.5$ ns, the snake sees the right cleaver come down at $x' = 125$ cm:



At $t' = 0$, the left cleaver comes down.
$\Rightarrow$ ~~If~~ the right cleaver does not rise, then the snake is hit on the head by the blunt side of the right cleaver! (But this happens in both frames.)

We can calculate this:
Frame S (cleaver frame):

length of snake in cleaver frame $= \frac{100\text{ cm}}{\gamma} = 80$ cm, $\quad \gamma = \frac{1}{\sqrt{1-(0.6)^2}} = 1.25$

Δt = time between chops $= 0$

In the snake frame, the right cleaver comes down at

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} 100\text{ cm} \\ \varnothing \end{pmatrix} = \begin{pmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{pmatrix} \begin{pmatrix} 100 \\ \varnothing \end{pmatrix}$$

$$\boxed{\Delta x' = 125\text{ cm}}$$

$$c\Delta t' = -75\text{ cm} \Rightarrow \boxed{\Delta t' = -2.5\text{ nano seconds}}$$

Some Formalism

Curious Fact: the quantity

$$(\Delta s)^2 \equiv (\Delta x)^2 - (c\Delta t)^2$$

has the same value in any frame of reference. We call Δs the "space-time interval" and we say that it is a "Lorentz Invariant"

Proof:

$$(\Delta x)^2 - (c\Delta t)^2 = (\gamma \Delta x' + \gamma\beta c\Delta t')^2 - (\gamma\beta\Delta x' + \gamma c\Delta t')^2$$

$$= \gamma^2(\Delta x' + \beta c\Delta t')^2 - \gamma^2(\beta\Delta x' - c\Delta t')^2$$

(cross terms now cancel)

$$= \underbrace{\gamma^2(1-\beta^2)}_{1}(\Delta x')^2 + \underbrace{\gamma^2(\beta^2-1)}_{-1}(c\Delta t')^2$$

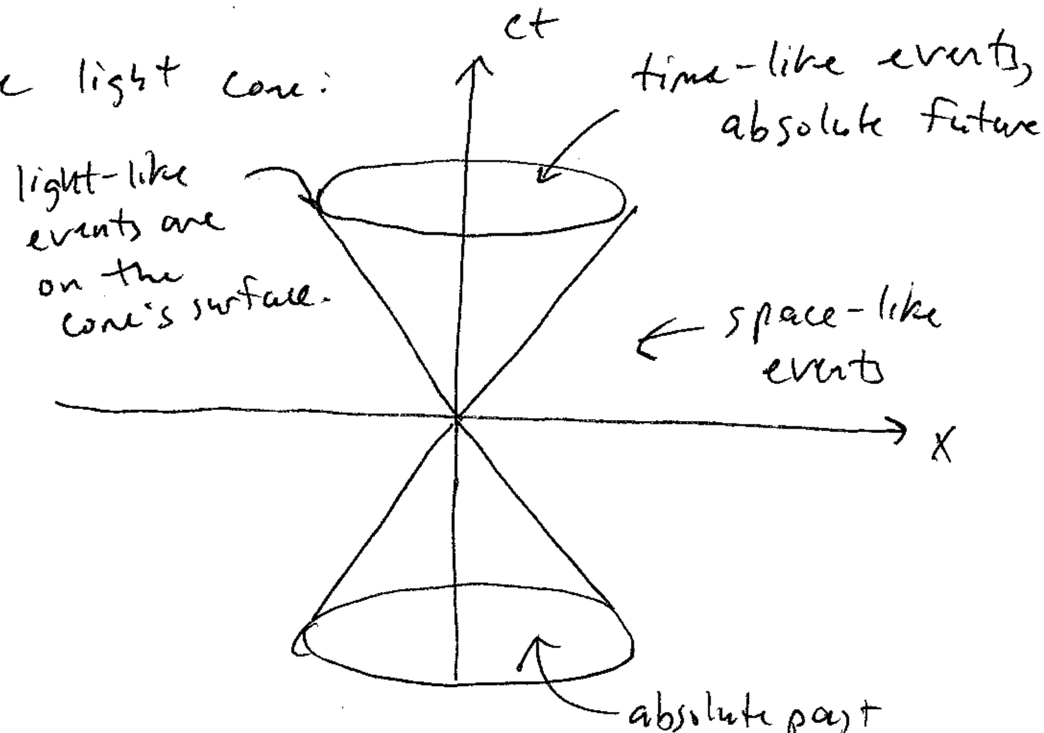
$$\boxed{= (\Delta x')^2 - (c\Delta t')^2}$$

So all observers agree on the numerical value of any (spacetime interval)^2

We categorize spacetime intervals as follows:

- $(\Delta s)^2 > 0 \Rightarrow$ "space-like" $\Rightarrow$ the two events are causally disconnected. Their order can be reversed by going to another frame of reference.
- $(\Delta s)^2 = 0 \Rightarrow$ "light-like"
- $(\Delta s)^2 < 0 \Rightarrow$ "time-like" $\Rightarrow$ the two events are causally related. Their order cannot be reversed by changing frames.

The light cone:



Formalism

We notice that the space-time interval Δs is calculated in a way that is similar to a dot product of a vector with itself:

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2$$

The only difference is that we use a $(-)$ sign for the $(c\Delta t)^2$ part rather than a $(+)$ sign. In fact if we include y & z, we have

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

Δy and Δz are the same in all reference frames, so $(\Delta s)^2$ is still a Lorentz Invariant.

We now define a new type of dot product that puts the $(-)$ sign in the correct place.

Let $\vec{X} = (x, y, z, ct)$

We want to have a dot product like this:

$$(\Delta s)^2 \equiv \vec{X} \cdot \vec{X} = x^2 + y^2 + z^2 - (c\Delta t)^2$$

So let's have 2 types of $\vec{x}$ vectors:

$$X = X_\mu = (x, y, z, ct) \leftarrow \text{"covariant 4-vector"}$$

$\mu = 1, 2, 3, 4$

$$X = X^\mu = (x, y, z, -ct) \leftarrow \text{"contravariant 4-vector"}$$

(-) sign!

Notice that when μ is a superscript (X^μ), the the vector has the (-) sign on the time component. Also notice the μ is a vector index running from 1 to 4:

$X_1 = x$	$X^1 = x$
$X_2 = y$	$X^2 = y$
$X_3 = z$	$X^3 = z$
$X_4 = ct$	$X^4 = -ct$

To take the dot product of 4-vector x with itself to calculate a space-time interval, we must always multiply X_μ by X^μ and sum over μ:

$$(\Delta s)^2 = \sum_{\mu=1}^{4} X_\mu X^\mu = xx + yy + zz + (ct)(-ct)$$
$$= x^2 + y^2 + z^2 - (ct)^2$$

we must always have one index upstairs (superscript) and one index downstairs (subscript) to take the dot product.

We now define a 2nd rank tensor (matrix) which changes a covariant vector to a contravariant vector:

$$g^{\mu\nu} \equiv \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & -1 \end{pmatrix} = \text{"metric tensor"}$$

Now we can convert between x_ν and x^μ:

$$x^\mu = \sum_{\nu=1}^{4} g^{\mu\nu} x_\nu$$

This means

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

or

$$x^1 = x_1$$
$$x^2 = x_2$$
$$x^3 = x_3$$
$$x^4 = -x_4 \quad \text{as desired.}$$

Similarly, we define

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & -1 \end{pmatrix}$$

With $g_{\mu\nu}$ we can do

$$x_\mu = \sum_{\nu=1}^{4} g_{\mu\nu} x^\nu$$

Notice that our notation becomes very clean if we use Einstein summation notation:

> Any repeated index, with one a superscript and one a subscript, implies a sum from 1 to 4.

Then $x_\mu = g_{\mu\nu} x^\nu$ (sum)

and $x^\mu = g^{\mu\nu} x_\nu$ (sum)

Then $(\Delta s)^2 = x \cdot x = \underbrace{x_\mu x^\mu}_{\text{sum}} = x_\mu (g^{\mu\nu} x_\nu)$ (sum, sum)

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$$

as desired.

Furthermore, let's define

$$g_\mu{}^\nu = g^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

one index up, and one index down!

(+) sign!

with this notation convention, moving any index from upstairs to downstairs or vice-versa has the effect of reversing the sign of the 4th component.

note that, by definition,

$$g_\mu{}^\nu = g^\mu{}_\nu = \delta_{\mu\nu}$$

Kronecker Delta.

4-vectors and the invariance of the scalar product

We will approach special relativity by re-writing all the familiar laws of Newtonian Mechanics in terms of 4-vectors such as (x, y, z, ct). If a law is written in terms of 4-vectors, then it explicitly complies with the requirements of special relativity, because 4-vectors transform in the correct way under a change of frame of reference.

We can determine a requirement on $g_{\mu\nu}$ and the Lorentz Transformation matrix by requiring that the scalar product of any 2 4-vectors is the same in any frame of reference. To see this relationship, let

A & B be 4-vectors, as measured in frame S.

In Frame S', A & B are called A' & B'. The Lorentz Transformation Matrix tells us how A' is related to A and how B' is related to B.

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{pmatrix} = \begin{pmatrix} \gamma & \emptyset & \emptyset & -\eta \\ \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ -\eta & \emptyset & \emptyset & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

and similarly for B' & B. Now let the ~~Rotation~~ Lorentz Transformation Matrix be called Λ (capital lambda). Thus

$A' = \Lambda A$ is the transformation of A

and $B' = \Lambda B$ is the transformation of B.

In summation notation:

$$\underbrace{A'^{\nu} = \Lambda^{\nu}{}_{\mu} A^{\mu}}_{\text{summation implied}} \quad \text{and} \quad \underbrace{B'^{\alpha} = \Lambda^{\alpha}{}_{\beta} B^{\beta}}_{\text{summation implied}}$$

Note that $\Lambda^{\nu}{}_{\mu}$ and $\Lambda^{\alpha}{}_{\beta}$ represent the <u>same matrix</u>. We give them different dummy indices because the implied sums are independent of each other.

The scalar product in S' is:

$$A' \cdot B' = \underbrace{A'_{\alpha} B'^{\alpha}}_{\text{summation implied}} = \underbrace{\left(g_{\alpha\nu} A'^{\nu}\right)}_{A'_{\alpha}} B'^{\alpha}$$

$$= g_{\alpha\nu} \underbrace{\left(\Lambda^{\nu}{}_{\mu} A^{\mu}\right)}_{A'^{\nu}} \underbrace{\left(\Lambda^{\alpha}{}_{\beta} B^{\beta}\right)}_{B'^{\alpha}}$$

Now we require that this scalar product be the same when calculated directly in frame S:

In S: $A \cdot B = A_{\beta} B^{\beta} = g_{\beta\mu} A^{\mu} B^{\beta}$

So we demand that

$$g_{\alpha\nu} \Lambda^{\nu}{}_{\mu} A^{\mu} \Lambda^{\alpha}{}_{\beta} B^{\beta} = g_{\beta\mu} A^{\mu} B^{\beta}$$

or

$$\left(\Lambda^\alpha{}_\beta\, g_{\alpha\nu}\, \Lambda^\nu{}_\mu\right)\underbrace{\left(A^\mu B^\beta\right)} = g_{\beta\mu}\underbrace{\left(A^\mu B^\beta\right)}$$

same

$$\therefore \quad \boxed{\Lambda^\alpha{}_\beta\, g_{\alpha\nu}\, \Lambda^\nu{}_\mu = g_{\beta\mu}}$$

In Matrix Notation, this reads as

$$\begin{pmatrix} \gamma & 0 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\eta & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\eta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

or

$$\boxed{\Lambda^T G \Lambda = G}$$

Note that this follows from $\gamma^2 - \eta^2 = 1$

when G is the matrix form of $g_{\mu\nu}$.

This equation says that the metric tensor $g_{\mu\nu}$ is unchanged under a Lorentz Transformation.

In fact, a better definition ~~at~~ of the Lorentz Transformation group is that it is the set of all matrices Λ that leave $g_{\mu\nu}$ unchanged