Is this a vector: $(1, 4, 2)$?

Maybe, but maybe not.

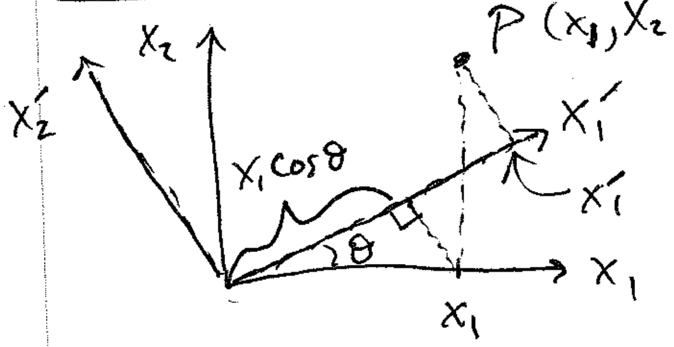
The defining characteristic of an ordinary 3-vector is the way it transforms under rotation. A vector should transform in the way that coordinates transform.

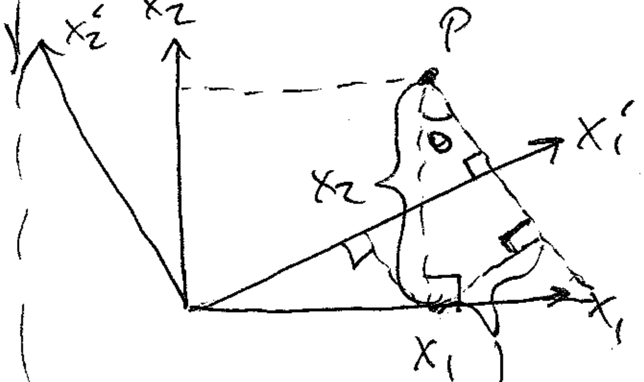
To decide if $(1, 4, 2)$ is a vector, we need to know how these three quantities transform under rotation.

Similarly, the defining characteristic of a scalar is that it remains invariant under rotation.

Strictly speaking, a vector is a "<u>tensor of the 1st rank</u>" while a scalar is a "<u>tensor of zeroth rank</u>".

Coordinate Transformations





$x_2 \sin\theta = x_2 \cos\left(\frac{\pi}{2} - \theta\right)$

$$\therefore x_1' = x_1 \cos\theta + x_2 \sin\theta = x_1 \cos\theta + x_2 \cos\left(\frac{\pi}{2} - \theta\right)$$

similarly, $x_2' = -x_1 \sin\theta + x_2 \cos\theta$

$$= x_1 \cos\left(\frac{\pi}{2}+\theta\right) + x_2 \cos\theta$$

Define $\lambda_{ij} \equiv \cos(x_i', x_j)$ Then

$$\lambda_{11} = \cos\theta$$
$$\lambda_{12} = \sin\theta$$
$$\lambda_{21} = -\sin\theta$$
$$\lambda_{22} = \cos\theta$$

Then
$$x_1' = \lambda_{11}x_1 + \lambda_{12}x_2$$
$$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2$$

In 3D,

$$x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3$$
$$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3$$
$$x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3$$

Better yet, use summation notation:

$$x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j \quad , i = 1, 2, 3$$ ← all 3 equations written at once.

(i) appears on left side, and is not summed on the right.

repeated index (j) indicates that we sum over (j)

Even better, do not write the limits of summation (assume the reader understands that we sum from one to three):

$$x_i' = \sum_j \lambda_{ij} x_j$$

Best notation: Einstein Notation: Any repeated index implies a summation:

$$x_i' = \lambda_{ij} x_j$$

← This is multiplication of a ~~a~~ 3×3 matrix by a vector

repeated (j) indicates that we sum over (j).

Clumsy notation: Matrix:

Let

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

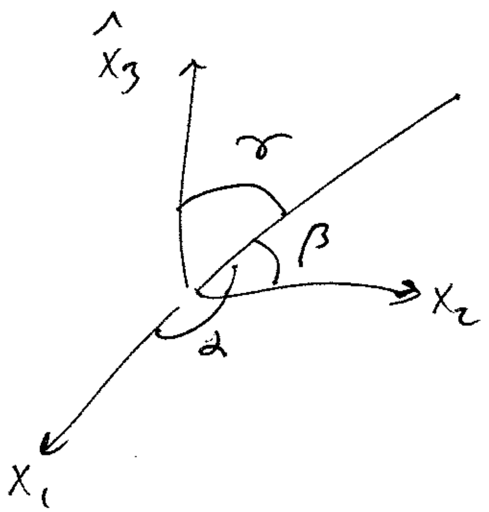
Then

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We call λ the rotation matrix. Also, sometimes we use $\underline{R}$ to represent the rotation matrix.

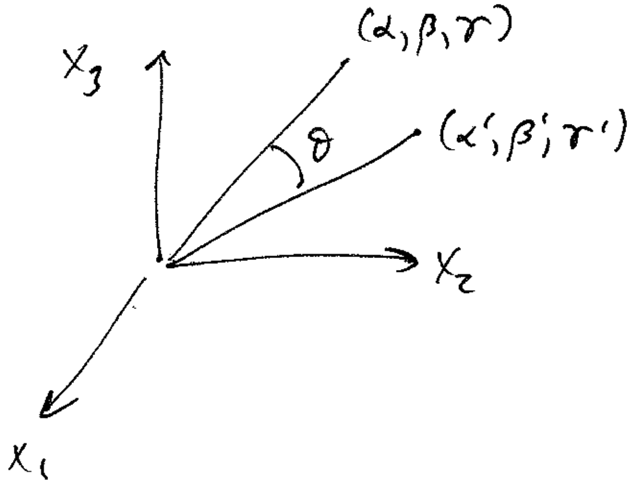
Properties of the Rotation Matrix

Let a line make angles α, β, γ with $\hat{x}_1, \hat{x}_2, \hat{x}_3$:



Trig identity:
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Consider a 2nd line which makes an angle θ with respect to the 1st line:



Then (trig identity)

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

Now consider the ~~###~~ 1^{st} line to be ~~as~~ the x_1' axis of a rotated coordinate system, and consider the 2^{nd} line to be the x_2' axis. We know that $x_1' \perp x_2'$, so $\cos\theta = \cos\frac{\pi}{2} = \emptyset$.

Also, $\cos\alpha = \lambda_{11}$, $\cos\alpha' = \lambda_{21}$, ect,

$\cos\beta = \lambda_{12}$, $\cos\beta' = \lambda_{22}$

$\cos\gamma = \lambda_{13}$, $\cos\gamma' = \lambda_{23}$

or $$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\theta = \emptyset$$

or $$\sum_j \lambda_{1j}\lambda_{2j} = \emptyset$$

In general $$\boxed{\sum_j \lambda_{ij}\lambda_{kj} = \emptyset, \quad i \neq k}$$

Three relationships between the $\{\lambda_{ij}\}$.

Also, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have
$$\sum_j \lambda_{1j}\lambda_{1j} = 1,$$

or, in general,

$$\boxed{\sum_j \lambda_{ij}\lambda_{kj} = 1, \quad i = k}$$

Three more relationships between the $\{\lambda_{ij}\}$.

We have 9 quanties $\{\lambda_{ij}\}$, but a total of six relationships, so 3 <u>degrees of freedom</u>.

These are the 3 rotation angles (in 3D).

We can write all six relationships as

$$\boxed{\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}}$$ ← orthogonality relation.

sum over 2nd indices

Kronecker Delta; $\delta_{ik} = 0,\ i \neq k$
$= 1,\ i = k$

By considering the (x) axes in the (x') coordinate system we would find

$$\boxed{\sum_i \lambda_{ij}\lambda_{ik} = \delta_{jk}}$$

sum over 1ST indices

<u>Transpose</u>: Define $\lambda^{+}_{ij} = \lambda_{ji}$

↑ reverse the order of indices.

Then consider the 2D case:

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

Now calculate $\lambda\lambda^{+} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}\begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}$

$$= \begin{pmatrix} (\lambda_{11})^2 + (\lambda_{12})^2 & (\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22}) \\ (\lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12}) & (\lambda_{21})^2 + (\lambda_{22})^2 \end{pmatrix}$$

But according to $\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}$ this is

$$\lambda\lambda^t = \begin{pmatrix} 1 & \emptyset \\ \emptyset & 1 \end{pmatrix} = \text{Identity matrix}$$

so $\boxed{\lambda^t \text{ is the inverse of } \lambda}$, i.e., multiplying λ by λ^t give $\underline{1}$.

$\boxed{\lambda^t = \lambda^{-1}}$ for <u>orthogonal</u> matrices.

In general rotations <u>do not commute</u>. Mathematically this is reflected in the fact that the rotation matrices do not commute. For example, a 90° rotation about x_3 is represented by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then a 90° rotation about ~~x_2~~ x_1 is

$$\lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The complete rotation, one after the other is

$$\vec{x}'' = \lambda_2 \lambda_1 \vec{x}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$$

But in the other order we have

$$\vec{x}''' = \lambda_1 \lambda_2 \vec{x}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \vec{x}$$ which is a different result.

Stated without proof: Any ordinary rotation matrix has determinant $= 1$. But if we reflect all three axis (changing from a right handed to a left-handed system), then the determinant will be -1.

Scalar Product: $\vec{A} \cdot \vec{B} = \sum_i A_i B_i$

Let's show that $\vec{A} \cdot \vec{B}$ does not change under a rotation, given that $A_i' = \sum_j \lambda_{ij} A_j$, $B_i' = \sum_k \lambda_{ik} B_k$

Then $\vec{A}' \cdot \vec{B}' = \sum_i A_i' B_i'$

$$= \sum_i \left(\sum_j \lambda_{ij} A_j \right) \left(\sum_k \lambda_{ik} B_k \right)$$

$$= \sum_{jk} \underbrace{\left(\sum_i \lambda_{ij} \lambda_{ik} \right)}_{\delta_{jk}} A_j B_k$$

$$= \sum_{jk} \delta_{jk} A_j B_k$$

$$= \sum_k A_k B_k$$

$$= \vec{A} \cdot \vec{B}$$

$\therefore \quad \vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$, so the dot product is a scalar (invariant).

Similarly, $|\vec{A}| \equiv \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{\sum_i x_i^2}$ is invariant (a scalar)

Unit vectors: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

Vector Product: Levi-Civita symbol (Fully antisymmetric)

$$\varepsilon_{ijk} \equiv \begin{cases} 1, & \text{if } (i,j,k) \text{ form an \underline{even} permutation of } (1,2,3) \\ -1, & \text{if } \cancel{ij}\, (i,j,k) \text{ form an \underline{odd} permutation} \\ \phi, & \text{if any of the } i,j,k \text{ are equal.} \end{cases}$$

Then $\varepsilon_{122} = \emptyset$,

$\varepsilon_{133} = \emptyset$, ect.

$\varepsilon_{123} = 1$

$\varepsilon_{132} = -1$

$\varepsilon_{312} = 1$

$\varepsilon_{321} = -1$, ect.

We defin the cross product as

$$\vec{C} = \vec{A} \times \vec{B}$$

where $\boxed{C_i \equiv \sum_{jk} \varepsilon_{ijk} A_j B_k}$

so $C_1 = \underbrace{\varepsilon_{123}}_{+1} A_2 B_3 + \underbrace{\varepsilon_{132}}_{-1} A_3 B_2$

$$C_1 = A_2 B_3 - A_3 B_2 \quad \text{as expected.}$$

Theorem: ε_{ijk} is related to δ_{ij} as follows:

$$\boxed{\sum_k \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

Can be checked by brute force for all possible i, j, l, m.

This allows us to prove identities such as

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Infintessimal Rotation: the $\vec{\omega}$ vector.

Rotate $\vec{r}$ through a very small angle about an arbitrary axis. The rotation matrix will be almost diagonal:

$$\vec{r}\,' = \lambda \vec{r} \approx (\underbrace{1}_{\text{identity matrix}} + \underbrace{\varepsilon}_{\text{a small correction matrix.}})\,\vec{r}$$

First note that infintessimal rotations do commute, even though finite rotations do not:

$$\underbrace{(1+\varepsilon_1)}_{\text{rotation matrix}}\underbrace{(1+\varepsilon_2)}_{\text{rotation matrix}} = (1)(1) + 1\varepsilon_2 + \varepsilon_1 1 + \underbrace{\varepsilon_1\varepsilon_2}_{\text{very small}}$$

$$\approx 1 + \varepsilon_1 + \varepsilon_2$$

while $(1+\varepsilon_2)(1+\varepsilon_1) = (1)(1) + \varepsilon_2 1 + 1\varepsilon_1 + \underbrace{\varepsilon_2\varepsilon_1}_{\text{very small}}$

$$\approx 1 + \varepsilon_1 + \varepsilon_2$$

$$\therefore \boxed{(1+\varepsilon_1)(1+\varepsilon_2) = (1+\varepsilon_2)(1+\varepsilon_1)}$$

Now let $\lambda = 1+\varepsilon$. We can show that $\lambda^{-1} = 1-\varepsilon$:

$$(1+\varepsilon)(1-\varepsilon) = 1 + \varepsilon - \varepsilon - \underbrace{\varepsilon\varepsilon}_{\text{very small}}$$

$$\approx 1$$

$$\therefore \boxed{\lambda^{-1} = 1-\varepsilon} \quad \text{where } \lambda = 1+\varepsilon$$

Also we know λ is an orthogonal matrix, $(\lambda^{-1} = \lambda^{+})$.

So, starting from $\lambda^{-1} = 1 - \varepsilon$ we have

$$(1+\varepsilon)^{-1} = 1-\varepsilon$$
$$(1+\varepsilon)^{+} = 1-\varepsilon$$
$$1^{+}+\varepsilon^{+} = 1-\varepsilon$$
$$\underline{1}+\varepsilon^{+} = \underline{1}-\varepsilon$$
$$\boxed{\varepsilon^{+} = -\varepsilon}$$

The transpose of ε is $-\varepsilon$, or ε is an anti-symmetric matrix. It must have this form:

$$\varepsilon = \begin{pmatrix} \emptyset & d\theta_3 & -d\theta_2 \\ -d\theta_3 & \emptyset & d\theta_1 \\ d\theta_2 & -d\theta_1 & \emptyset \end{pmatrix}$$ for some set of $d\theta_1, d\theta_2, d\theta_3$.

Now we can see how $\vec{r}$ transforms under the infintessimal rotation:

$$\vec{r}\,' = (1+\varepsilon)\vec{r} = \vec{r} + \varepsilon\vec{r}$$
$$\vec{r}\,' - \vec{r} = \varepsilon\vec{r}$$
$$d\vec{r} = \varepsilon\vec{r}$$

In matrix form:

$$d\vec{r} = \begin{pmatrix} \emptyset & d\theta_3 & -d\theta_2 \\ -d\theta_3 & \emptyset & d\theta_1 \\ d\theta_2 & -d\theta_1 & \emptyset \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} d\theta_3 y - d\theta_2 z \\ d\theta_1 z - d\theta_3 x \\ d\theta_2 x - d\theta_1 y \end{pmatrix}$$

or $\quad d\vec{r} = \vec{r} \times d\vec{\theta} \quad$ where $d\vec{\theta} = (d\theta_1, d\theta_2, d\theta_3)$

($d\vec{\theta}$ is a legitimate vector because infintessimal rotations commute. A finite rotation is not a vector however, because finite rotations do not commute.)

Thus $\quad \dfrac{d\vec{r}}{dt} = \vec{r} \times \underbrace{\dfrac{d\vec{\theta}}{dt}}_{\vec{\omega}}$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{r} \times \vec{\omega}}$$

So our angular velocity vector $\vec{\omega}$ is related to the rotation matrix:

$$\lambda = \begin{pmatrix} 1 & \omega_3 dt & -\omega_2 dt \\ -\omega_3 dt & 1 & \omega_1 dt \\ \omega_2 dt & -\omega_1 dt & 1 \end{pmatrix}$$

Transformation Properties of the Inertia Tensor

$\vec{L}$ = angular momentum, $\vec{\omega}$ = angular velocity, I = Inertia tensor, and we have

$$\vec{L} = I\vec{\omega} \quad \text{or} \quad L_k = \sum_l I_{kl}\omega_l$$ ← summation notation

Now because $\vec{L}$ and $\vec{\omega}$ are vectors, we know how to project this equation onto a different set of coordinate axes (because we know how vectors transform under rotation.)

First, in the primed coordinate system the equation looks like

$$L'_i = \sum_j I'_{ij}\omega'_j$$

We'd like to understand the relationship between I and I', that is, how the inertia tensor transforms from one coordinate system to another. Since we know how $\vec{L}$ and $\vec{\omega}$ transform, we can determine how I transforms:

$$L_k = \sum_m \lambda_{mk} L'_m \quad \text{and} \quad \omega_l = \sum_j \lambda_{jl}\omega'_j$$

(because $\vec{L}$ and $\vec{\omega}$ are vectors)

Therefore

$$\sum_m \lambda_{mk} L'_m = \sum_l I_{kl}\left(\sum_j \lambda_{jl}\omega'_j\right)$$

Now multiply both sides by λ_{ik} and sum over (k):
(this is matrix multiplication)

$$\sum_m \underbrace{\left(\sum_k \lambda_{ik}\lambda_{mk}\right)}_{\delta_{im}} L'_m = \sum_j \left(\sum_{k,l} \lambda_{ik}\lambda_{jl} I_{kl}\right)\omega'_j$$

because λ is an orthogonal matrix.
So sum the left side over (m):

$$L'_i = \sum_j \left(\sum_{kl} \lambda_{ik}\lambda_{jl} I_{kl}\right)\omega'_j$$

We also know that in the primed system we have

$$L'_i = \sum_j I'_{ij}\omega'_j$$

So now we can identify what I'_{ij} looks like:

$$I'_{ij} = \sum_{kl} \lambda_{ik}\lambda_{jl} I_{kl}$$

reverse indices, $\lambda_{jl} = \lambda^+_{lj}$

or

$$\boxed{I'_{ij} = \sum_{kl} \lambda_{ik} I_{kl} \lambda^+_{lj}}$$

In Matrix notation this says

$$\boxed{I' = \lambda I \lambda^{+}}$$

or $$\boxed{I' = \lambda I \lambda^{-1}}$$

This is the rule for how to transform I under a coordinate rotation: given the rotation matrix λ, you sandwich I between λ and λ^{+} to get the new I. This is called a similarity transformation.

In fact, we call any ~~2 dim~~ 3×3 matrix (of the 2nd rank) which transforms according to this rule a tensor. And the inertia tensor gets its name from the fact that it does transform in this way.

Diagonalization of the inertia tensor

If we can guess a rotation matrix which transforms a non-diagonal I into a diagonal I, then the axes of the new coordinate system are the principal axes.

<u>Example</u> Show that if the principal moments λ_1, λ_2, and λ_3 are all equal, then <u>any</u> direction in space is a principal axis:

we have
$$I = I_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_0 \mathbb{1}$$

Making an arbitrary coordinate transformation (λ), we have
$$I' = \lambda I \lambda^{-1} = I_0 \lambda \mathbb{1} \lambda^{-1}$$
$$= I_0 \lambda \lambda^{-1}$$
$$\boxed{= I_0 \mathbb{1}}$$

So the inertia tensor is diagonal in the rotated system, no matter what rotation we make.

<u>In summary</u>

- <u>A tensor of the zeroth rank</u> is a <u>scalar</u>.

  $\Rightarrow$ It does not transform under a rotation of axes: $\alpha = \alpha'$

- A tensor of the <u>1st rank</u> is a <u>vector</u>.

  $\Rightarrow$ It transforms as
  $$A_i' = \sum_j \lambda_{ij} A_j$$

- A tensor of the <u>2nd rank</u> is a type of matrix.

  $\Rightarrow$ It transforms as $I_{ij}' = \sum_{kl} \lambda_{ik} I_{kl} \lambda_{lj}^{\dagger}$