Three Formulations of Classical Mechanics.

(1) Newtonian: (1687): objects accelerate in response to forces. To fully analyze a system, the forces must be known.

(2) Lagrangian (1788): The Equations of Motion can be determined from the "Lagrangian" ~~Function~~, which is the difference between the Kinetic Energy and the Potential Energy:

$$L = T - U$$

The Lagrangian formulation rests on Hamilton's Principle, which is a variation principle, and the Calculus of Variations.

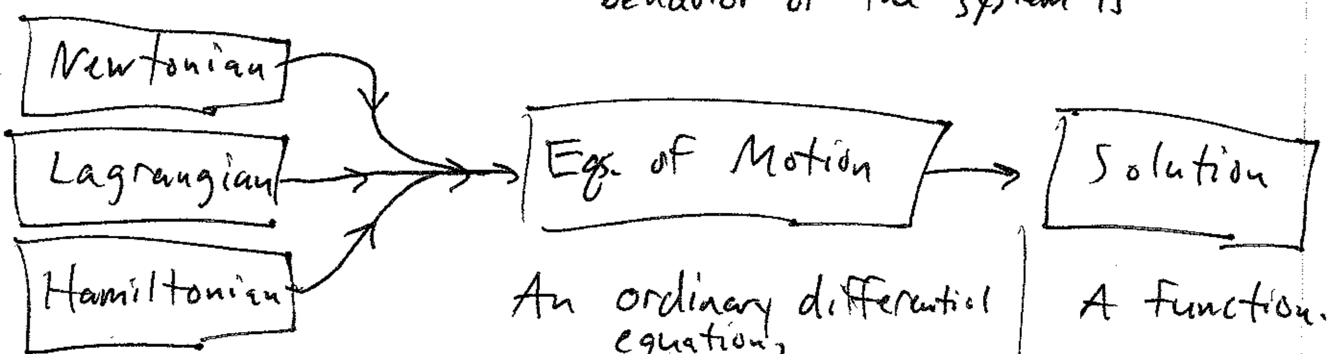
(3) Hamiltonian (1833): The Eqs. of Motion can be determined from the "Hamiltonian" which is often the sum of Kinetic and Potential Energy:

$$H = T + U \quad (\text{in many cases})$$

The Hamiltonian formulation is built on top of Lagrangian Mechanics.

- In Phys 410 we will study and use all three formulations of Classical Mechanics.

- The three formulations are alternative methods of finding an equation of motion. Ultimately all three methods should agree on what the ~~solution~~ ~~equation of motion is.~~ behavior of the system is



An ordinary differential equation,
<u>or</u> a set of ODEs,
<u>or</u> a partial differential equation

A function.

- The three formulations give us different insights into why the system behaves the way it does.
    - Newtonian forces are closest to our own experiences, and perhaps give us the greatest intuitive understanding.
    - The Lagrangian and Hamiltonian formulations give us better insight into conserved quantities and symmetries.

- Which formulation we use is mostly determined by which is most convenient or illuminating.
    - The Newtonian method is best when there are drag forces or dissipation.
    - The Lagrangian and Hamiltonian methods are usually ~~easiest~~ best when there are forces of constraint, like the normal force between a sliding block and an inclined frictionless ramp, or between the cars of a roller coaster and the rails. Often we do not care about the details of the normal forces or constraint forces, and the Lagrangian and Hamiltonian methods allow us to ignore them.
    - The Hamiltonian Formalism is the one that is best suited for providing a basis for quantum mechanics, and other types of modern physics like plasmas and non-linear dynamics.

In Phys 410 we will primarily use Newtonian and Lagrangian methods. We will spend less time on Hamiltonian Mechanics.

Differential Equations

For systems of particles, the equation of motion is one or more ordinary differential equations. (ODE).

For continuous systems like fluids or elastic media, the equations of motion is a partial differential equation

We will be studying systems of particles for the most part.

(1) Finding the equation of motion is about 50% of the work.

(2) Solving the equation of motion and studying the solution is the other 50%. However, this part of the job should be straightforward, because we should always be able to solve any differential equation.

How to solve ODEs.

(1) Guess the answer (often this works)

(2) Use a method from an ODEs math class or textbook. (if (1) did not work)

③ Better yet, solve it numerically.

Because any ODE can be solved numerically, method ③ is the most general, and perhaps the best one. We will use all 3 methods.

What you need to know to solve an ODE:

1) First try guessing a solution. Then move on to ② and ③

2) An ODE with a maximum of (n) order derivatives has (n) free parameters and requires (n) initial conditions for a complete and unique solution.

→ For example, since Newton's 2nd Law specifies the acceleration ($\ddot{x}$), we need 2 initial condition to solve for $x(t)$ (x_0 and v_0, for example).

3) A few ODEs are ~~[illegible]~~ so simple and common that you must know their solutions right away. They are:

(a) $$\frac{dy}{dt} = A \xrightarrow{\text{sol'n}} y(t) = At + c$$

↑ one free parameter.

(b) $\frac{dy}{dt} = Ay \overset{\text{Soln}}{\Longrightarrow} y(t) = c\, e^{At}$

$\uparrow$ one free parameter.

(c) $\frac{d^2y}{dt^2} + Ay = \phi$ (Harmonic Oscillator).

Many ways to write the solution (all equivalent)

$$y(t) = c_1 \sin(\sqrt{A}\,t + c_2)$$
$$y(t) = c_1 \cos(\sqrt{A}\,t + c_2)$$
$$y(t) = c_1 e^{i(\sqrt{A}t + c_2)}$$
$$y(t) = c_1 \sin(\sqrt{A}\,t) + c_2 \cos(\sqrt{A}\,t)$$
$$y(t) = c_1 \sinh(\sqrt{-A}\,t) + c_2 \cosh(\sqrt{-A}\,t)$$
$$y(t) = c_1 \sinh(\sqrt{-A}\,t + c_2)$$
$$y(t) = c_1 \cosh(\sqrt{-A}\,t + c_2)$$

4) The Euler Method for numerical solutions to ODEs is easy to do. Since

$$y'(t) \approx \frac{y(t+\Delta t) - y(t)}{\Delta t},$$

then $y(t+\Delta t) \approx y(t) + (\Delta t)\, y'(t)$.

Numerically, let
$$t \to t_i$$
$$t+\Delta t \to t_{i+1}$$
$$y(t) \to y_i$$
$$y(t+\Delta t) \to y_{i+1}.$$

Then $y_{i+1} = y_i + y'_i \Delta t$.

This tells us how to step along in time and find $y(t)$ as long as we know $y'(t)$. The method is more stable if we can use y'_{i+1} instead of y'_i:

$$y_{i+1} = y_i + y'_{i+1} \Delta t \ .$$

In mechanics we often have an equation of motion which tells us how to calculate the acceleration from all the other variables. For example:

$$\ddot{x} = -\omega^2 x$$

or

$$a = -\omega^2 x \ .$$

To solve numerically, we write

$$v_{i+1} = v_i + a_{i+1} \Delta t$$

and $x_{i+1} = x_i + v_{i+1} \Delta t$

At each step we calculate $a_{i+1} = -\omega^2 x_i$ which is approximate, because $x_i \approx x_{i+1}$. Then given x_0 and v_0, we can solve $\ddot{x} = -\omega^2 x$ for all time.

Even better would be to use a more accurate method like Runge-Kutta

5) In 1-D single particle Newtonian Mechanics, the Eq. of Motion is often something like:

$$m\ddot{x} = F(t, x, v)$$

Three cases to know:

<u>(a) F is a function of t only (F(t)):</u>

$\Rightarrow$ Just integrate twice to get x(t):

$$m\frac{dv}{dt} = F(t)$$

$$m\,dv = F(t)\,dt$$

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')\,dt'$$

$$m\left(v(t) - v_0\right) = \int_{t_0}^{t} F(t')\,dt'$$

$$\boxed{v(t) = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t')\,dt'}$$

Then repeat:

$$\frac{dx}{dt} = v(t)$$

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\,dt'$$

$$x(t) - x_0 = \int_{t_0}^{t} v(t')\,dt'$$

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'}$$

(b) F is a Function of x only: $(F(x))$:

We use the following trick to re-write accel.:

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

Then $F = ma$ becomes $F(x) = mv\frac{dv}{dx}$

Now separate variables and integrate:

$$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')\,dx'$$

$$m\left[\frac{(v(x))^2}{2} - \frac{v_0^2}{2}\right] = \int_{x_0}^{x} F(x')\,dx'$$

$$\boxed{v(x) = \left[v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'\right]^{\frac{1}{2}}}$$

Then, since $\frac{dx}{dt} = v(x)$, we have

$$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt' = t - t_0$$

$$\boxed{t = t_0 + \int_{x_0}^{x(t)} \frac{dx'}{v(x')}}$$

This yields (t) as a function of (x). In principle you might be able to rewrite it as (x) as a function of (t). If not, then (x) can be solved for numerically for any given value of (t).

(c) F is a function of (v) only $(F(v))$

We have $m\dfrac{dv}{dt} = F(v)$

$$\text{or} \quad m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt' = t - t_0$$

$$\text{or} \quad \boxed{t = t_0 + m\int_{v_0}^{v(t)} \frac{dv'}{F(v')}}$$

This gives t as a function of v. In principle you can solve for v, yielding $v(t)$. Then integrate $v(t)$ to get $x(t)$:

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'}$$

Or, if we desire to obtain $v(x)$, we can take this route:

$$a = v\frac{dv}{dx} = \frac{F(v)}{m}$$

$$m \int_{v_0}^{v(t)} \frac{v' dv'}{F(v')} = \int_{x_0}^{x} dx' = x - x_0$$

$$\boxed{x = x_0 + m \int_{v_0}^{v(t)} \frac{v' dv'}{F(v')}}$$

This gives x as a function of v, which can be inverted if desired.

<u>Vector Analysis</u> (briefly)

Various Notations for vectors:

$$\vec{r} = (x, y, z)$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$, $\hat{x}, \hat{y}, \hat{z}$ are unit vectors

$$\vec{r} = x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3$$

$$\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3$$, where $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$

The last one is the best because it allows us to use <u>Summation Notation</u>

$$\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3 = \underbrace{\sum_i r_i\hat{e}_i}_{\text{elegant.}}$$

Dot product: $\vec{r}\cdot\vec{s} \equiv r_1 s_1 + r_2 s_2 + r_3 s_3$

$$= \underbrace{\sum_i r_i s_i}$$

Summation Notation (elegant).

Cross product: $\vec{r}\times\vec{s} = (r_y s_z - r_z s_y)\hat{x}$
$+ (r_z s_x - r_x s_z)\hat{y}$
$+ (r_x s_y - r_y s_x)\hat{z}$

$$= \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{pmatrix}$$

determinant

Differentiation, $\dfrac{d\vec{r}}{dt} = \lim_{\Delta t\to 0} \dfrac{\Delta\vec{r}}{dt}$

Then $\dfrac{d}{dt}(\vec{r}+\vec{s}) = \dfrac{d\vec{r}}{dt} + \dfrac{d\vec{s}}{dt}$

and $\dfrac{d}{dt}(F\vec{r}) = \dfrac{dF}{dt}\vec{r} + F\dfrac{d\vec{r}}{dt}$

Also $\dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{x} + \dfrac{dy}{dt}\hat{y} + \dfrac{dz}{dt}\hat{z} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$

Newton's Laws

1) In the absence of forces, a particle moves with constant velocity $\vec{v}$.

$\Rightarrow$ This is true in inertial frames of reference. We define an inertial frame as one in which the 1st Law holds true.

2) $$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} = m\ddot{\vec{r}}$$

Also, defining momentum as $\vec{p} \equiv m\vec{v}$, we can write

$$\dot{\vec{p}} = m\dot{\vec{v}} = m\vec{a} = \vec{F}$$

or $$\boxed{\vec{F} = \dot{\vec{p}}}$$

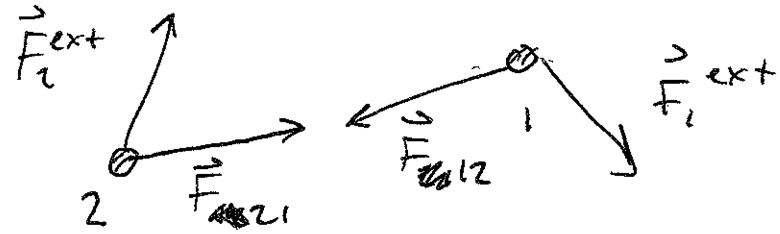
3) If object ① exerts force $\vec{F}_{21}$ on object ②, then object ② always exerts a reaction force $\vec{F}_{12}$ on object ①. The reaction force is equal in magnitude and opposite in direction to $\vec{F}_{21}$:

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

This leads to conservation of momentum.

To see how conservation of momentum arises, consider a 2-particle case.
Let ① exert force on ② and ② on ① as well.

In addition, let ① and ② experience external forces $\vec{F}_1^{ext}$ and $\vec{F}_2^{ext}$.



Then $\dot{\vec{p}}_1 = \vec{F}_{12} + \vec{F}_1^{ext}$

and $\dot{\vec{p}}_2 = \vec{F}_{21} + \vec{F}_2^{ext}$

Then let $\vec{P} \equiv$ total momentum $\equiv \vec{p}_1 + \vec{p}_2$.

Then
$$\dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \vec{F}_1^{ext} + \vec{F}_2^{ext} + \underbrace{\vec{F}_{12} + \vec{F}_{21}}_{\text{cancel, according to the 3rd Law.}}$$

Then $\dot{\vec{P}} = \vec{F}_1^{ext} + \vec{F}_2^{ext} \equiv \vec{F}^{ext}$ = total external force.

In particular, if $\vec{F}^{ext} = \phi$, then $\dot{\vec{P}} = \phi$, and $\vec{P}$, the total momentum, is constant.

A similar argument holds for multi-particle systems.

Newton's 2nd Law in cartesian coordinates.

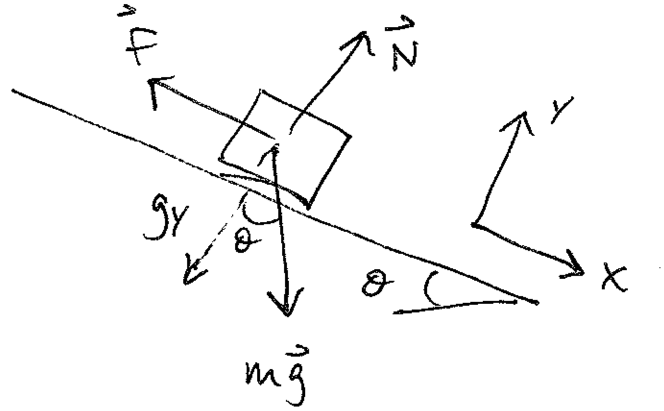
$$\vec{F} = m\ddot{\vec{r}} = m\left(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$$

$$\left(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}\right)$$

so

$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y}$$
$$F_z = m\ddot{z}$$

Example: Block sliding down ramp with friction.



Use tilted coordinate system so all motion is in the x-direction.

Then $F_y = |\vec{N}| - mg_y$

$= |\vec{N}| - mg\cos\theta = \emptyset$ since the block does not accelerate in y direction.

Therefore $|\vec{N}| = mg\cos\theta$.

This is helpful because friction is proportional to $\vec{N}$:

$$F_x = \mu N = \mu mg\cos\theta$$

So the x-direction reads as

$$mg_x - F_x = m\ddot{x}$$

$$mg_x - \mu mg\cos\theta = m\ddot{x}$$

$$g_x = g\sin\theta$$

$$\boxed{g\sin\theta - \mu g\cos\theta = \cancel{m}\ddot{x}}$$

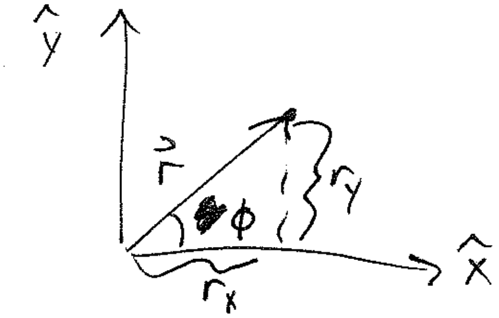
Then, if the block is released from rest so that $v_0 = 0$, then

$$v(t) = g(\sin\theta - \mu\cos\theta)t$$

and $\boxed{x(t) = \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2}$

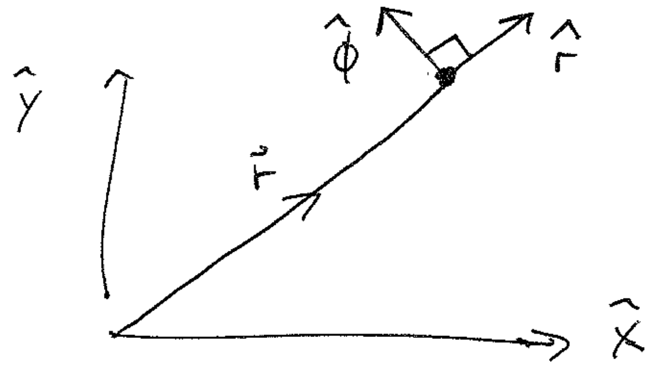
if $x_0 = 0$ as well.

Two-Dimensional Polar Coordinates

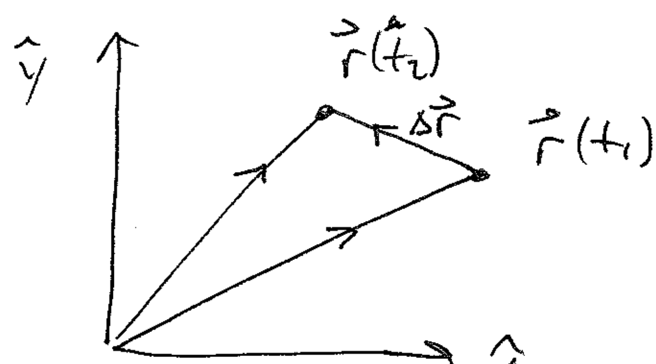


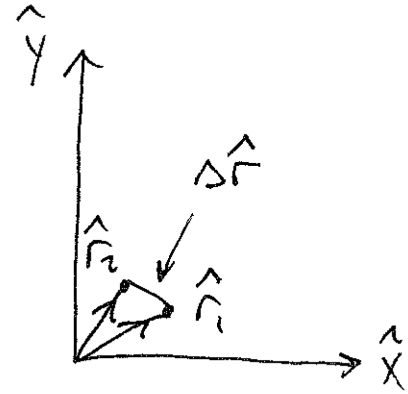
$$r_x = |\vec{r}| \cos\phi = x \qquad r = \sqrt{x^2 + y^2}$$
$$r_y = |\vec{r}| \sin\phi = y \qquad \phi = \tan^{-1}(y/x)$$

We can define and use unit vectors $\hat{r}$ & $\hat{\phi}$ which move as $\vec{r}$ moves in the plane.



$\hat{r}$ points towards $\vec{r}$ and $\hat{\phi}$ is $\perp$ to $\hat{r}$. Since $\hat{r}$ & $\hat{\phi}$ move, we must take their time derivative when we differentiate $\vec{r}$:





$\Delta\hat{r}$ is in the direction of $\hat{\phi}$:

$$\Delta\hat{r} \approx \Delta\phi\,\hat{\phi}$$

$$\approx \frac{\Delta\phi}{\Delta t}(\Delta t)\,\hat{\phi}$$

$$\Delta\hat{r} \approx \dot{\phi}\,\Delta t\,\hat{\phi}$$

so that $\boxed{\dfrac{d\hat{r}}{dt} = \dot{\phi}\,\hat{\phi}}$

Then $\dot{\vec{r}} = \dfrac{d}{dt}(r\hat{r}) = \dfrac{dr}{dt}\hat{r} + r\dfrac{d\hat{r}}{dt}$

$$\boxed{\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\,\hat{\phi}}$$

Then $v_r = \dot{r}$ and $v_\phi = r\dot{\phi} = r\omega$

where $\dot{\phi} \equiv \omega$.

To get the acceleration, differentiate again.

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{\vec{r}}) = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\phi}\hat{\phi})$$

The rule for the derivative of $\hat{\phi}$ is

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\,\hat{r}.$$

Then $\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dfrac{d\hat{\phi}}{dt}$

$$\boxed{\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}}$$

For pure circular motion, where $\dot r = \ddot r = 0$, then

$$\ddot{\vec r} = \vec a = -r\dot\phi^2\,\hat r + r\ddot\phi\,\hat\phi$$

or $$\vec a = -r\omega^2\,\hat r + r\alpha\,\hat\phi$$

where ω = angular velocity and

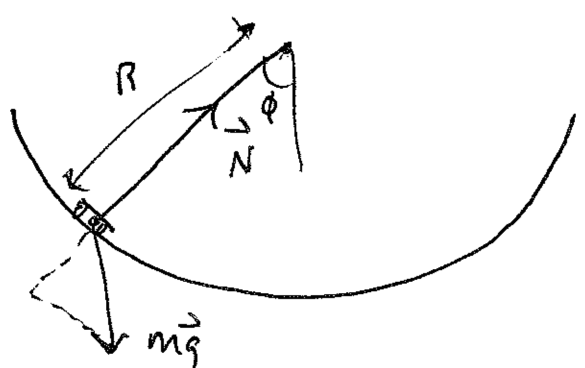
α = angular acceleration.

Now Newton's 2nd Law in 2D polar coordinates is:

$$F_r = m(\ddot r - r\dot\phi^2)$$

$$F_\phi = m(r\ddot\phi + 2\dot r\dot\phi)$$

Skateboard in a semi-circular half-pipe.



Then $F_\phi = \boxed{-mg\sin\phi = mR\ddot\phi}$

$F_r = mg\cos\phi - N$

ϕ Eq. of Motion: $\boxed{\ddot\phi = -\frac{g}{R}\sin\phi}$

This Equation of Motion cannot be solved in terms of elementary functions. It can be solved numerically.

However, if ϕ is small (and remains small), then

$$\sin\phi \approx \phi \quad \text{and}$$

$$\boxed{\ddot{\phi} + \frac{g}{R}\phi \approx 0}$$ Simple Harmonic oscillator.

Air Resistance

Model the drag force due to air resistance:

$$F(v) = \underbrace{bv}_{\text{linear term}} + \underbrace{cv^2}_{\text{quadratic term}}$$

In many cases only one of the terms is important. For $v \approx 0$, the linear term is more important, and for $v =$ large, the quadratic term is more important.

Linear air resistance

A projectile with linear drag:

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v}$$
$$m\dot{\vec{v}} = m\vec{g} - b\vec{v}$$

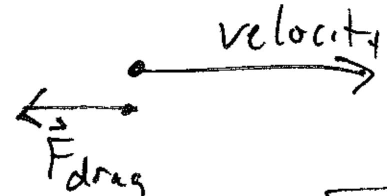
drag force is always opposite $\vec{v}$.

Components:

$$m\dot{v}_x = -bv_x$$
$$m\dot{v}_y = mg - bv_y$$

← position (y) measured vertically downward, so $g > \emptyset$ and in the y direction.

Horizontal case (no gravity):



No gravity: $\boxed{\dot{v}_x = \frac{-b}{m} v_x}$ $\overset{\text{solution}}{\Longrightarrow}$ $\boxed{v_x(t) = Ae^{-kt},}$ $k \equiv b/m$

A is a free parameter determined by initial conditions (initial velocity in this case.):

At $t = \emptyset$, $v_x(t=\emptyset) = A$, so $A = v_{x0}$

(Final Solution for velocity:) $v_x(t) = v_{x0} e^{-kt}$, $k = b/m$

Integrate $v_x(t)$ to get $x(t)$:

$$v_x = \frac{dx}{dt}$$

$$\int_{t_0=\emptyset}^{t} v_x(t')\,dt' = \int_{x_0}^{x(t)} dx' = x(t) - x_0$$

$$\Downarrow$$

$$\int_{t_0=\emptyset}^{t} v_{x0} e^{-kt'}\,dt'$$

$$\Downarrow$$

$$-\frac{v_{x0}}{k} e^{-kt'} \Big|_{t_0=\emptyset}^{t}$$

$$\Downarrow$$

$$-\frac{v_{x0}}{k}\left(e^{-kt} - 1\right) = x(t) - x_0$$

$$x(t) = x_0 + \frac{v_{x0}}{k}\left(1 - e^{-kt}\right)$$

or, if we define $\tau \equiv \frac{1}{k}$ = units of seconds,

then

$$x(t) = x_0 + v_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

~~that~~ As $t \to \infty$, $x \to v_{x0}\tau$ (assuming x_0 is small).

We can call $v_{x0}\tau \equiv x_\infty$

Then $x(t) = x_0 + x_\infty (1 - e^{-t/\tau})$

If we measure x from x_0, then $x_0 = \emptyset$ and

$$\boxed{x(t) = x_\infty (1 - e^{-t/\tau})}$$

Vertical case (linear drag with gravity):

Again, measure positive (y) downward, so $g > \emptyset$:

$$m\dot{v}_y = mg - bv_y$$

positive

Always opposite v

It's possible for the total force to be zero, when $|mg| = |bv_y|$. Then the acceleration is zero, so v_y is constant. This is the terminal velocity:

$$\cancel{v_{ter}}\quad mg - bv_{ter} \equiv \emptyset$$

$$\boxed{v_{ter} = \frac{mg}{b}}$$

v_{ter} is positive, because downward is positive.

Now rewrite the Eq. of Motion in terms of v_{ter}:

$$m\dot{v}_y = -b\left(\frac{-mg}{b} + v_y\right)$$

$$\boxed{m\dot{v}_y = -b(v_y - v_{ter})}$$

We can solve easily be changing velocity variables to measure velocity with respect to the terminal velocity:

$u \equiv v_y - v_{ter}$. Thus $\dot{u} = \dot{v}_y$, because $\dot{v}_{ter} = \emptyset$.

$$m\dot{u} = -b(u)$$

$$\dot{u} = \frac{-b}{m} u \Rightarrow \text{soln: } \boxed{u(t) = Ae^{-t/\tau},}$$

$$\tau \equiv \frac{m}{b}$$

or $v_y(t) - v_{ter} = Ae^{-t/\tau}$

What is A? ~~Ae~~ Set $t = \emptyset$ to determine A in terms of initial y velocity (v_{y0}):

At $t = \emptyset$, $\underbrace{v_y(t=\emptyset)}_{v_{y0}} - v_{ter} = A$

$$\boxed{A = v_{y0} - v_{ter}}$$

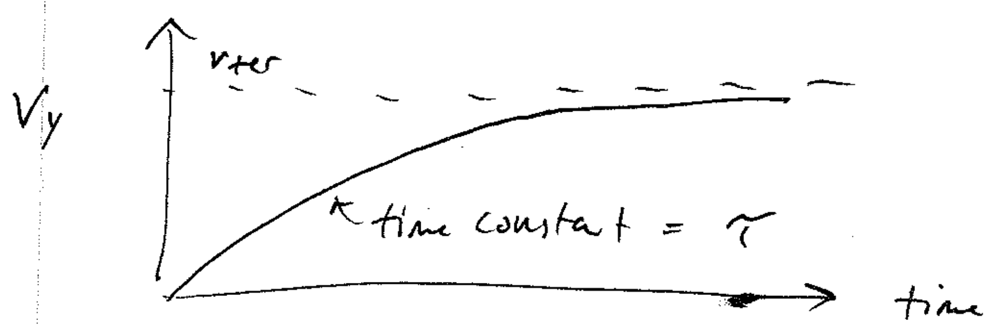
$$v_y(t) = v_{ter} + (v_{y0} - v_{ter})e^{-t/\tau}$$

or
$$\boxed{v_y(t) = v_{y0}e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)}$$

As expected, $v_y \to v_{ter}$ as $t \to \infty$.

If dropped from rest, $v_{y0} = \emptyset$, then we have

$$v_y(t) = v_{ter}\left(1 - e^{-t/\tau}\right)$$



To get $y(t)$, integrate $v_y(t)$:

$$y(t) = y_0 + \int_0^t v_y(t')\,dt'$$

$$= y_0 + \int_0^t \left(v_{ter} + (v_{y0} - v_{ter})e^{-t'/\tau}\right)dt'$$

$$\boxed{y(t) = y_0 + v_{ter}t + (v_{y0} - v_{ter})\tau\left(1 - e^{-t/\tau}\right)}$$

Quadratic Drag

~~Now~~ Now we assume v is large enough that quadratic drag is a better model:

$$m\ddot{\vec{r}} = m\vec{g} - cv^2\hat{v} = m\vec{g} - cv\vec{v}$$

drag always opposite $\vec{v}$ & $\hat{v}$

measure (+)y downward so $\vec{g}$ is (+).

In components:

$$m\dot{v}_x = -c\left(\sqrt{v_x^2+v_y^2}\right)(v_x)$$

$$m\dot{v}_y = mg - c\left(\sqrt{v_x^2+v_y^2}\right)v_y$$

Two coupled ODEs, non-linear.

These coupled non-linear ODEs cannot be solved analytically, but they may be solved numerically.

On the other hand, for pure horizontal motion, $v_y = \phi$ and $\dot{v}_y = \phi$, then we have only one equation to solve:

Pure Horizontal Motion, Quadratic Drag:

- $$m\frac{dv}{dt} = -cv^2$$

Using our trick: (seperation of variables):

$$\frac{m\,dv}{v^2} = -c\,dt$$

$$m\int_{v_0}^{v(t)} \frac{dv'}{v'^2} = -c\int_{\emptyset}^{t} dt' = -c(t - \xrightarrow{\emptyset}t_0)$$

$$\Downarrow$$

$$m\left(-\frac{1}{(v(t))} + \frac{1}{v_0}\right)$$

$$\Downarrow$$

$$m\left(\frac{1}{v_0} - \frac{1}{v(t)}\right) = -c(t - \xrightarrow{\emptyset}t_0) = -ct$$

Solve for $v(t)$:

$$v(t) = \left(\frac{1}{v_0} + \frac{ct}{m}\right)^{-1} = \frac{1}{\frac{1}{v_0} + \frac{ct}{m}} = \frac{v_0}{1 + \frac{cv_0 t}{m}}$$

$\frac{m}{cv_0}$ has units of time, so define

$$\tau \equiv \frac{m}{cv_0}. \text{ Then } \boxed{v(t) = \frac{v_0}{1 + t/\tau}}$$

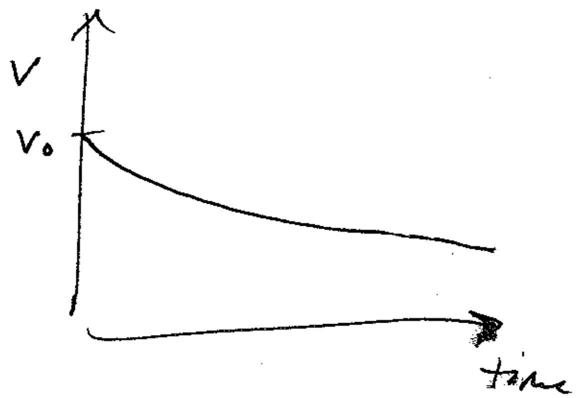
To find $x(t)$, integrate $v(t)$:

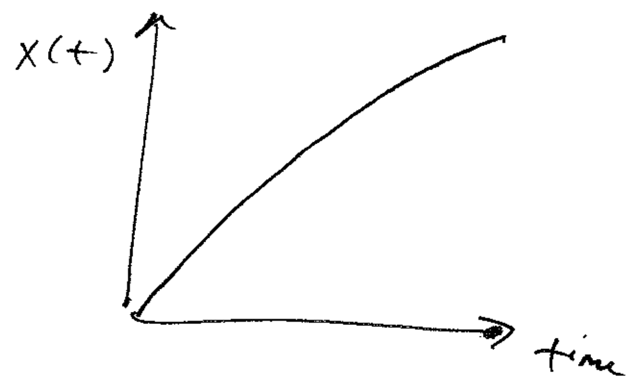
$$x(t) = x_0 + \int_0^t v(t')\,dt'$$

let $x_0 = \emptyset$

$$\boxed{x(t) = v_0 \tau \ln\left(1 + t/\tau\right)} \quad \text{(ignoring } x_0\text{)}$$

The velocity approaches zero slowly:





And $x(t)$ grows forever. But we have assumed the drag is always quadratic, no matter how small v is. In an experiment, eventually v is small enough that linear drag is important, and then v will decay exponentially.

Vertical Motion with quadratic drag.

$$m\dot{v} = mg - cv^2$$

terminal velocity: when $mg - cv_{ter}^2 = \emptyset$

$$v_{ter} = \sqrt{\frac{mg}{c}}$$

Re-write Eq. of motion by substituting

$$c = \frac{mg}{v_{ter}^2} \qquad \text{Then}$$

$$\boxed{\dot{v} = g\left(1 - \frac{v^2}{v_{ter}^2}\right)} \qquad \text{Eq. of Motion.}$$

$$\frac{dv}{1 - v^2/v_{ter}^2} = g\,dt$$

$$\underbrace{\int_{v_0}^{v(t)} \frac{dv'}{1 - v'^2/v_{ter}^2}} = g\int_0^t dt' = g(t - \emptyset)$$

$$v_{ter}\tanh^{-1}\left(\frac{v}{v_{ter}}\right) = t$$

$\left(\tanh \text{ is hyperbolic tangent, } \tanh(x) = \frac{\sinh(x)}{\cosh(x)}\right)$

Solve for v:

$$\boxed{v(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)}$$

Integrate to get y(t):

$$\boxed{y(t) = \frac{v_{ter}^2}{g} \ln\left(\cosh\left(\frac{gt}{v_{ter}}\right)\right) + y_0}$$