This is a brief introduction to the ideas and concepts of nonlinear mechanics, and a discussion of various quantitative methods for analyzing such problems.

We will focus on the driven damped pendulum (DDP)

\[ \ddot{\phi} + \frac{b}{m} \dot{\phi} + \frac{g}{L} \sin \phi = \frac{F_0}{mL} \cos \omega t. \]

![Diagram of driven damped pendulum](image)

**Figure 12.1** The three important forces on the driven damped pendulum are the resistive force with magnitude \( bv \), the weight \( mg \), and the driving force \( F(t) \). (There is also a reaction force from the pivot at the top, but this contributes nothing to the torque.)
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\[
\begin{align*}
    m\ddot{x} &= -kx \\
    mL^2\ddot{\varphi} &= -mgL \sin \varphi \\
    m\ddot{\mathbf{r}} &= -\frac{G M m}{r^2}\mathbf{r}
\end{align*}
\]
Linear

Nonlinear

Chaotic

Linear
Driven, Damped Pendulum (DDP)

\[ mL^2 \ddot{\phi} = -mg \sin \phi - bL^2 \dot{\phi} + LF(t) \]

| Driving force: \( F(t) = F_0 \cos(\omega t) \) |
| Damping constant: \( \beta = \frac{b}{2m} \) |
| Natural frequency: \( \omega_0 = \sqrt{\frac{g}{L}} \) |
| Drive strength: \( \gamma = \frac{F_0}{mg} \) |

\[ \ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \sin(\phi) = \gamma \omega_0^2 \cos(\omega t) \]

We expect something interesting to happen as \( \gamma \to 1 \), i.e. the driving force becomes comparable to the weight.
A Route to Chaos
NDsolve in Mathematica

\[ \text{Fig12p2} = \text{NDSolve}\left[\{\phi'[t] + 2 \beta \phi'[t] + \omega_0^2 \sin[\phi[t]] = \gamma \omega_0^2 \cos[\omega t], \phi[0] = 0, \phi'[0] = 0\}, \phi, \{t, 0, 6\}\right] \]

\[ \{\phi \mapsto \text{InterpolatingFunction}\left[\{\{0., 6.\}\}, <>\}\right]\} \]

\[ \text{Plot}\left[\text{Evaluate}[\phi[t] /\text{Fig12p2}], \{t, 0, 6\}, \text{Ticks} \rightarrow \{\{1, 2, 3, 4, 5, 6\}, \{-0.3, 0.3\}\}, \text{PlotRange} \rightarrow \text{All}, \right. \]

\[ \left. \text{AxesStyle} \rightarrow \text{Thick}, \text{PlotStyle} \rightarrow \text{Thick}, \text{LabelStyle} \rightarrow \{\text{Bold, Medium}\}\right] \]
Driven, Damped Pendulum (DDP)

\[ \ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \sin(\phi) = \gamma \omega_0^2 \cos(\omega t) \]

For all following plots:

\[ \omega = 2\pi \]

Period = \( 2\pi/\omega = 1 \)

\[ \omega_0 = 1.5\omega \]

\[ \beta = \omega_0 / 4 \]

\[ \phi(0) = \dot{\phi}(0) = 0 \]
Small Oscillations of the Driven, Damped Pendulum

$\gamma \ll 1$ will give small oscillations

Linear

$$\ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \phi = \gamma \omega_0^2 \cos(\omega t)$$

$\gamma = 0.2$

After the initial transient dies out, the solution looks like

$$\phi(t) = A \cos(\omega t - \delta)$$

Periodic “attractor”
Small Oscillations of the Driven, Damped Pendulum

\( \gamma << 1 \) will give small oscillations

1) The motion approaches a unique periodic attractor independent of initial conditions

2) The motion is sinusoidal with the same frequency as the drive

\[ \phi(t) = A \cos(\omega t - \delta) \]
Moderate Oscillations of the Driven, Damped Pendulum

$\gamma < 1$ and the nonlinearity becomes significant…

$$\ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \left( \phi - \frac{1}{6} \phi^3 \right) \approx \gamma \omega_0^2 \cos(\omega t)$$

Try $\phi(t) = A \cos(\omega t - \delta)$

This solution gives from the $\phi^3$ term: $\cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x)$

Since there is no $\cos(3\omega t)$ on the RHS, it must be that $\phi, \dot{\phi}, \ddot{\phi}$ all develop a $\cos(3\omega t)$ time dependence. Hence we expect:

$$\phi(t) = A \cos(\omega t - \delta) + B \cos[3(\omega t - \delta)]$$

$B \ll A$

We expect to see a third harmonic as the driving force grows
Harmonics

- Frequency = $n\omega$
Moderate Driving: The Nonlinearity Distorts the $\cos(\omega t)$

The motion is periodic, but ...

The third harmonic distorts the simple $\phi(t) = A \cos(\omega t - \delta)$
Even Stronger Driving: Complicated Transients – then Periodic!

After a wild initial transient, the motion becomes periodic

$$\gamma = 1.06$$

After careful analysis of the long-term motion, it is found to be periodic with the same period as the driving force.
Slightly Stronger Driving: Period Doubling

After a wilder initial transient, the motion becomes periodic, but period 2!

The long-term motion is TWICE the period of the driving force!

A SUB-Harmonic has appeared

\[ \gamma = 1.073 \]
Harmonics and Subharmonics

- Frequency = \( \frac{\omega}{n} \)

- Frequency = \( n\omega \)

\( n \) is an integer
Slightly Stronger Driving: Period 3

The period-2 behavior still has a strong period-1 component. Increase the driving force slightly and we have a very strong period-3 component.

\[ \gamma = 1.077 \]

Period 3
Multiple Attractors

The linear oscillator has a single attractor for a given set of initial conditions

For the drive damped pendulum:
Different initial conditions result in different long-term behavior (attractors)

\[ \phi(t) \]
\[ \gamma = 1.077 \]

- \( \phi(0) = \dot{\phi}(0) = 0 \) (Period 3)
- \( \phi(0) = -\pi / 2, \dot{\phi}(0) = 0 \) (Period 2)
Period Doubling Cascade

\[ \gamma = 1.06 \]

Early-time motion

\[ \gamma = 1.078 \]

Close-up of steady-state motion

\[ \gamma = 1.081 \]

\[ \gamma = 1.0826 \]

\[ \phi(0) = -\pi / 2 \]

\[ \dot{\phi}(0) = 0 \]
Period Doubling Cascade

$\gamma = 1.06$

$\gamma = 1.078$

$\gamma = 1.081$

$\gamma = 1.0826$
‘Bifurcation Points’ in the Period Doubling Cascade

Driven Damped Pendulum

\[ \phi(0) = -\pi / 2 \]
\[ \dot{\phi}(0) = 0 \]

\[ \delta = 4.6692016 \] is called the Feigenbaum number

The spacing between consecutive bifurcation points grows smaller at a steady rate:

\[ (\gamma_{n+1} - \gamma_{n}) \approx \frac{1}{\delta} (\gamma_{n} - \gamma_{n-1}) \]

\[ \approx \rightarrow = \text{ as } n \rightarrow \infty \]

The limiting value as \( n \rightarrow \infty \) is \( \gamma_c = 1.0829 \). Beyond that is … chaos!
Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of $\gamma$.

Such period-doubling cascades are seen in many nonlinear systems. Their form is essentially the same in all systems – it is “universal.”
Period infinity
Chaos!

The pendulum is “trying” to oscillate at the driving frequency, but the motion remains erratic for all time.
Chaos

- Nonperiodic
- Sensitivity to initial conditions
Sensitivity of the Motion to Initial Conditions

Start the motion of two identical pendulums with slightly different initial conditions. Does their motion converge to the same attractor? Does it diverge quickly?

Two pendulums $\phi_1(t), \phi_2(t)$ are given different initial conditions.

Follow their evolution and calculate $\Delta \phi(t) = \phi_2(t) - \phi_1(t)$

For a linear oscillator $\phi(t) = A \cos(\omega t - \delta) + C_1 e^{\gamma t} + C_2 e^{\gamma^2 t}$

$\Delta \phi(t) = D e^{-\beta t} \cos(\omega_1 t - \delta_1)$

Thus the trajectories will converge after the transients die out.

The initial conditions affect the transient behavior, the long-term attractor is the same.

Long-term attractor

Transient behavior

$r_{1,2} = -\beta \pm i\omega_1$ underdamping, $\omega_1 = \sqrt{\omega^2 - \beta^2}$
Convergence of Trajectories in Linear Motion

\[ \Delta \phi(t) = De^{-\beta t} \cos(\omega_1 t - \delta_1) \]

Take the logarithm of \(|\Delta \phi(t)|\) to magnify small differences.

\[ \ln(|\Delta \phi(t)|) = \ln(D) - \beta t + \ln(|\cos(\omega_1 t - \delta_1)|) \]

Plotting \(\log_{10}[|\Delta \phi(t)||]\) vs. \(t\) should be a straight line of slope \(-\beta\), plus some wiggles from the \(\ln[|\cos(\omega_1 t - \delta_1)||]\) term

Note that \(\log_{10}[x] = \log_{10}[e] \ln[x]\)
Convergence of Trajectories in Linear Motion

\[ \log_{10}[|\Delta \phi(t)|] \]

\[ \gamma = 0.1 \]
\[ \Delta \phi(0) = 0.1 \text{ Radians} \]

The trajectories converge quickly for the small driving force (\(\sim\) linear) case.
This shows that the linear oscillator is essentially insensitive to its initial conditions!
Convergence of Trajectories in Period-2 Motion

The trajectories converge more slowly, but still converge.

\[
\log_{10}[|\Delta \phi(t)|]
\]

\[
\gamma = 1.07
\]

\[
\Delta \phi(0) = 0.1 \text{ Radians}
\]
Divergence of Trajectories in Chaotic Motion

Log$_{10}||\Delta \phi(t)||$

$\gamma = 1.105$
$\Delta \phi(0) = 0.0001$ Radians

The trajectories diverge, even when very close initially $\Delta \phi(16) \sim \pi$, so there is essentially complete loss of correlation between the pendulums.

Extreme Sensitivity to Initial Conditions
Practically impossible to predict the motion

If the motion remains bounded, as it does in this case, then $\Delta \phi$ can never exceed $2\pi$. Hence this plot will saturate.
The Lyapunov Exponent

\[ |\Delta \phi(t)| \sim Ke^{\lambda t} \quad K > 0 \]

\[ \lambda = \text{Lyapunov exponent} \]

\[ \lambda < 0: \text{ periodic motion in the long term} \]

\[ \lambda > 0: \text{ chaotic motion} \]
<table>
<thead>
<tr>
<th>Linear</th>
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<th>Chaos</th>
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<tbody>
<tr>
<td>Drive period</td>
<td>Harmonics,</td>
<td>Nonperiodic,</td>
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<tr>
<td>$\lambda &lt; 0$</td>
<td>Subharmonics,</td>
<td>Extreme sensitivity</td>
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<td></td>
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$\lambda < 0$
What Happens if we Increase the Driving Force Further?
Does the chaos become more intense?

$\gamma = 1.13$

$\Delta \phi(0) = 0.001$ Radians

Period 3 motion re-appears!

With increasing $\gamma$ the motion alternates between chaotic and periodic
What Happens if we Increase the Driving Force Further? Does the chaos re-appear?

$\Delta \phi(0) = 0.001$ Radians

Chaotic motion re-appears!

This is a kind of ‘rolling’ chaotic motion
Divergence of Two Nearby Initial Conditions for Rolling Chaotic Motion

\[ \phi(t) \quad \gamma = 1.503 \quad t \]

\[ \Delta \phi(0) = 0.001 \text{ Radians} \]

Chaotic motion is always associated with extreme sensitivity to initial conditions

Periodic and chaotic motion occur in narrow intervals of \( \gamma \)
Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of $\gamma$

Such period-doubling cascades are seen in many nonlinear systems. Their form is essentially the same in all systems – it is “universal”.

Driven Diode experiment

F₀ cos(ωt)

Sub-harmonic frequency spectrum

FIG. 1. Experimental apparatus for subharmonic generation.

FIG. 5. Power spectral density (dB) vs frequency for $f = 98$ kHz, dynamic range 70 dB, showing subharmonics to $f/32$. The components agree with prediction (dashed bars, Ref. 14) within 2 dB rms deviation, except for the peak at $f/16$.
Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of $\gamma$

Such period-doubling cascades are seen in many nonlinear systems. Their form is essentially the same in all systems – it is “universal”.

Figure 3.7. Period doubling in seizures and in thermoconvection. (A), intracranial recording from a patient suffering temporal lobe seizures. The left-hand side recordings show how the length of the period, marked by the rectangles, doubles during the seizure, while the right-hand side recording corresponds to a posterior time toward the end of the seizure where more complex aperiodic activity is apparent due to successive period doublings. (B), to illustrate how period doubling was defined in classic studies and the similarity with the intracerebral recording, doubling of the periods in a thermoconvection experiment is shown. The time series is the temperature of a fluid, note the subharmonic cascade as the control parameter of the system changes. The line segments indicate the duration of one period. Panel A is reprinted with permission from Perez Velazquez et al. (2003) and panel B from Bergé et al. (1984).

The Brain-behaviour Continuum: The Subtle Transition Between Sanity and Insanity
By Jose Luis. Perez Velazquez

Fig. 12.9, Taylor
Bifurcation Diagram

Used to visualize the behavior as a function of driving amplitude $\gamma$

1) Choose a value of $\gamma$
2) Solve for $\phi(t)$, and plot a periodic sampling of the function $\phi(t_0), \phi(t_0+1), \phi(t_0+2), \phi(t_0+3), \phi(t_0+4),...$
time $t_0$ chosen at a time after the attractor behavior has been achieved
3) Move on to the next value of $\gamma$

$$\phi(0) = -\pi/2$$
$$\dot{\phi}(0) = 0$$
Construction of the Bifurcation Diagram

- $\gamma = 1.06$
- $\gamma = 1.078$
- $\gamma = 1.081$
- $\gamma = 1.0826$

Period 1
Period 2
Period 4
Period 8

Period 6 window

$\phi(t)$

$t = 501, 502, \ldots, 600$
The Rolling Motion Renders the Bifurcation Diagram Useless

\[ \gamma = 1.503 \]

As an alternative, plot \( \dot{\phi}(t) \)

\[ \Delta \phi(0) = 0.001 \text{ Radians} \]
Bifurcation Diagram Over a Broad Range of $\gamma$

- Previous diagram range
- Mostly chaos
- Period-3
- Mostly chaos
- Period-1 followed by period doubling bifurcation
- Mostly chaos

$\phi(t)$

Rolling Motion (next slide)
Period-1 Rolling Motion at $\gamma = 1.4$

Even though the pendulum is “rolling”, $\dot{\phi}(t)$ is periodic
An Alternative View: State Space Trajectory

Plot $\dot{\phi}(t)$ vs. $\phi(t)$ with time as a parameter

$\phi(0) = -\pi / 2$
$\dot{\phi}(0) = 0$

$\gamma = 0.6$

First 20 cycles

Cycles 5 - 20

Fig. 12.20, 12.21
An Alternative View: State Space Trajectory

Plot $\dot{\phi}(t)$ vs. $\phi(t)$ with time as a parameter

$\gamma = 0.6$

$\phi(0) = 0$

$\dot{\phi}(0) = 0$

The periodic attractor: $[\phi(t),\dot{\phi}(t)]$ is an ellipse

$$\phi(t) = A \cos(\omega t - \delta)$$

$$\dot{\phi}(t) = -A \omega \sin(\omega t - \delta)$$

The state space point moves clockwise on the orbit

Fig. 12.22
State Space Trajectory for Period Doubling Cascade

$\gamma = 1.078$

$\gamma = 1.081$

Period-2

Period-4

Plotting cycles 20 to 60

Fig. 12.23
State Space Trajectory for Chaos

\( \gamma = 1.105 \)

The orbit has not repeated itself…
State Space Trajectory for Chaos

\[ \gamma = 1.5 \]
\[ \beta = \frac{\omega_0}{8} \]

Chaotic rolling motion
Mapped into the interval \(-\pi < \phi < \pi\)

This plot is still quite messy. There’s got to be a better way to visualize the motion …
The Poincaré Section

Similar to the bifurcation diagram, look at a sub-set of the data

1) Solve for $\phi(t)$, and construct the state-space orbit
2) Plot a periodic sampling of the orbit

$$\begin{bmatrix} \phi(t_0), \dot{\phi}(t_0) \\ \phi(t_0 + 1), \dot{\phi}(t_0 + 1) \\ \phi(t_0 + 2), \dot{\phi}(t_0 + 2) \end{bmatrix}, \ldots$$

with $t_0$ chosen after the attractor behavior has been achieved

$\gamma = 1.5$
$\beta = \omega_0/8$
The Poincaré Section is a Fractal

The Poincaré section is a much more elegant way to represent chaotic motion.
The Superconducting Josephson Junction as a Driven Damped Pendulum

\[ \Psi_1 = \Psi_1 e^{i\phi_1}, \quad \Psi_2 = \Psi_2 e^{i\phi_2} \]

\[ \phi = \phi_2 - \phi_1 = \text{phase difference of SC wave-function across the junction} \]

\[ I_{dc} + I_{rf} \cos(\omega t) = \frac{\hbar C}{2e} \left( \ddot{\phi} + \frac{1}{RC} \dot{\phi} + \frac{2eI_0}{\hbar C} \sin \phi \right) \]
Radio Frequency (RF) Superconducting Quantum Interference Devices (SQUIDs)

\[ \Phi_{\text{applied}} + \Phi_{\text{induced}} = n\Phi_0 \]

\[ \Phi_{DC} + \Phi_{rf} \sin \omega t = \frac{\Phi_0 \delta}{2\pi} + L \left( I_C \sin \delta + \frac{L \Phi_0}{R} \frac{d\delta}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\delta}{dt^2} \right) \]

Flux Quantization in the loop

\[ \delta = \theta_1 - \theta_2 - \frac{2e}{\hbar} \int_1^2 \vec{A} \cdot d\vec{l} \]

\[ 2\pi \left[ f_{DC} + f_{rf} \sin \left( \frac{\omega}{\omega_0} \tau \right) \right] = \delta + \beta_{rf} \sin \delta + \frac{1}{RC\omega_0} \frac{d\delta}{d\tau} + \frac{d^2\delta}{d\tau^2} \]

\[ \beta_{rf} = \frac{2\pi LI_C}{\Phi_0} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \tau = \omega_0 t \]
Single rf-SQUID  
THz Emission from the Intrinsic Josephson Effect
A classic problem in nonlinear physics

\[ \frac{d\phi}{dt} = \frac{2\pi}{\Phi_0} V \]

\[ I = I_c \sin \phi \]

\[ \Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-15} \text{Tm}^2 \]

\[ f_{JJ} = \frac{2e}{h} V = (0.483 \text{THz/mV})V \]

DC voltage on junction creates an oscillating \( \phi(t) \), which in turn creates an AC current that radiates

Best emission is seen when the crystal is partially heated above \( T_c \).
Results are extremely sensitive to details (number of layers, edge properties, type of material, width of mesa, etc.).
Many competing states do not show emission

Emission enhanced near cavity mode resonances \( \rightarrow \) requires non-uniform current injection, assisted by inhom.
heating, \( \pi \)-phase kinks, crystal defects

Chaos in Newtonian Billiards

Imagine a point-particle trapped in a 2D enclosure and making elastic collisions with the walls.

Describe the successive wall-collisions with a “mapping function”

\[ s_{n+1} = f(s_n, \theta_n) \]
\[ \theta_{n+1} = g(s_n, \theta_n) \]

Linear Maps for “Integrable” systems!!

Non-Linear Maps for “Chaotic” systems!!

- The “Chaos” arises due to the shape of the boundaries enclosing the system.

Computer animation of extreme sensitivity to initial conditions for the stadium billiard