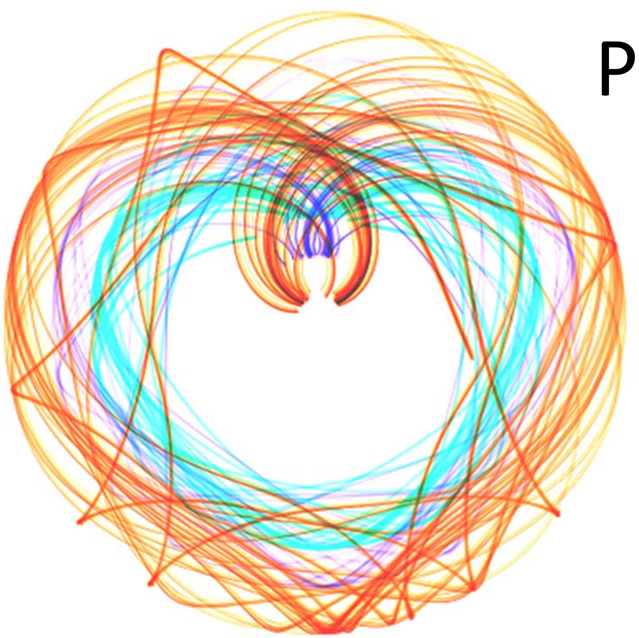
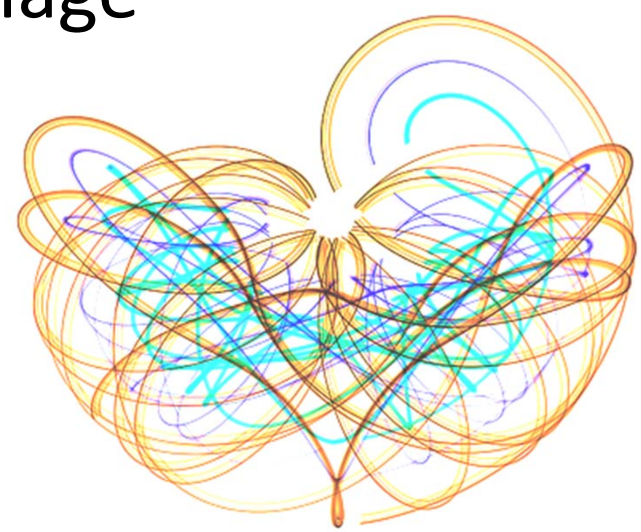
12

Nonlinear Mechanics and Chaos

Dr. Jen-Hao Yeh

Prof. Anlage





This is a brief introduction to the ideas and concepts of nonlinear mechanics, and a discussion of various quantitative methods for analyzing such problems

We will focus on the driven damped pendulum (DDP)

$$\ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi = \frac{F_0}{mL}\cos\omega t.$$

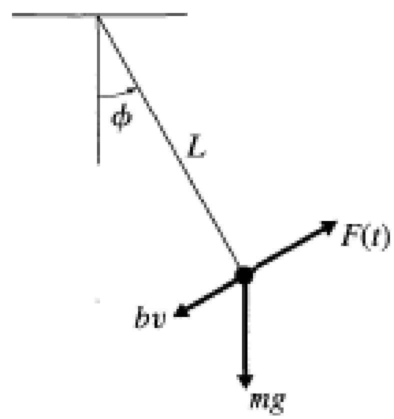


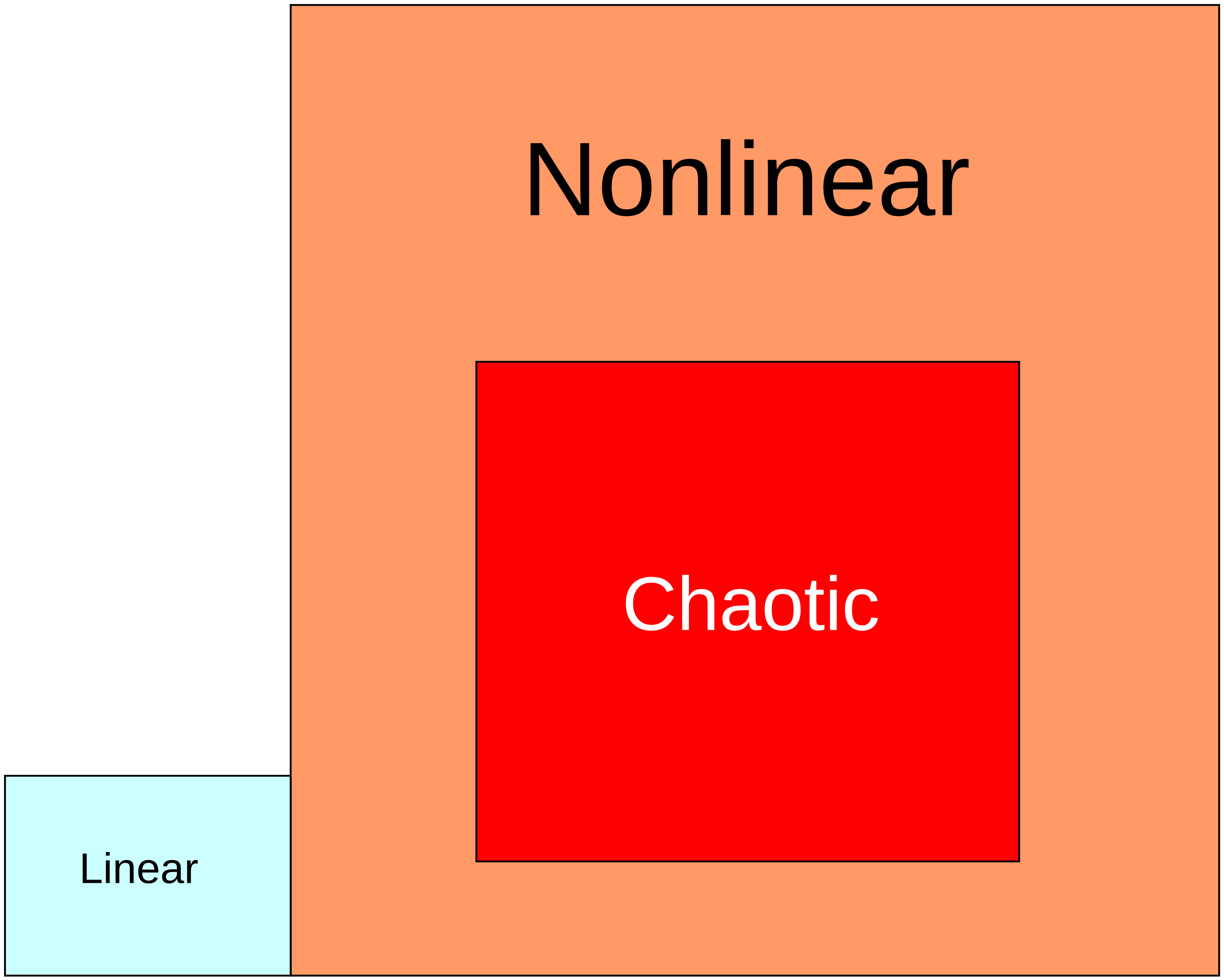
Figure 12.1 The three important forces on the driven damped pendulum are the resistive force with magnitude bv, the weight mg, and the driving force $F(t)$. (There is also a reaction force from the pivot at the top, but this contributes nothing to the torque.)

Linear	Nonlinear
easy	hard
special	general
analytical	numerical
Superposition principle	
	Chaos

$$m\ddot{x} = -kx$$

$$mL^2\ddot{\varphi} = -mgL\sin\varphi$$

$$m\ddot{\vec{r}} = -\frac{GMm}{r^2}\hat{r}$$


Nonlinear
Chaotic
Linear

Driven, Damped Pendulum (DDP)

$$mL^2\ddot{\phi} = -mg\sin\phi - bL^2\dot{\phi} + LF(t)$$

Driving force: $F(t) = F_0\cos(\omega t)$

Damping constant: $\beta = \frac{b}{2m}$

Natural frequency: $\omega_0 = \sqrt{\frac{g}{L}}$

Drive strength: $\gamma = \frac{F_0}{mg}$

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\sin(\phi) = \gamma\omega_0^2\cos(\omega t)$$

We expect something interesting to happen as $\gamma \to 1$,
i.e. the driving force becomes comparable to the weight


A Route to Chaos

NDSolve in Mathematica

```
Fig12p2 = NDSolve[{ϕ''[t] + 2 β ϕ'[t] + ω0^2 Sin[ϕ[t]] == γ ω0^2 Cos[ω t], ϕ[0] == 0, ϕ'[0] == 0}, ϕ, {t, 0, 6}]
```

```
{{ϕ → InterpolatingFunction[{{0., 6.}}, <>]}}
```

```
Plot[Evaluate[ϕ[t] /. Fig12p2], {t, 0, 6}, Ticks → {{1, 2, 3, 4, 5, 6}, {-0.3, 0.3}}, PlotRange → All,
    AxesStyle → Thick, PlotStyle → Thick, LabelStyle → {Bold, Medium}]
```

Driven, Damped Pendulum (DDP)

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin(\phi) = \gamma\omega_0^2 \cos(\omega t)$$

For all following plots:

$$\boxed{\begin{gathered} \omega = 2\pi \\ \textbf{Period} = \mathbf{2\pi/\omega = 1} \\ \omega_0 = 1.5\omega \\ \beta = \omega_0 / 4 \end{gathered}}$$

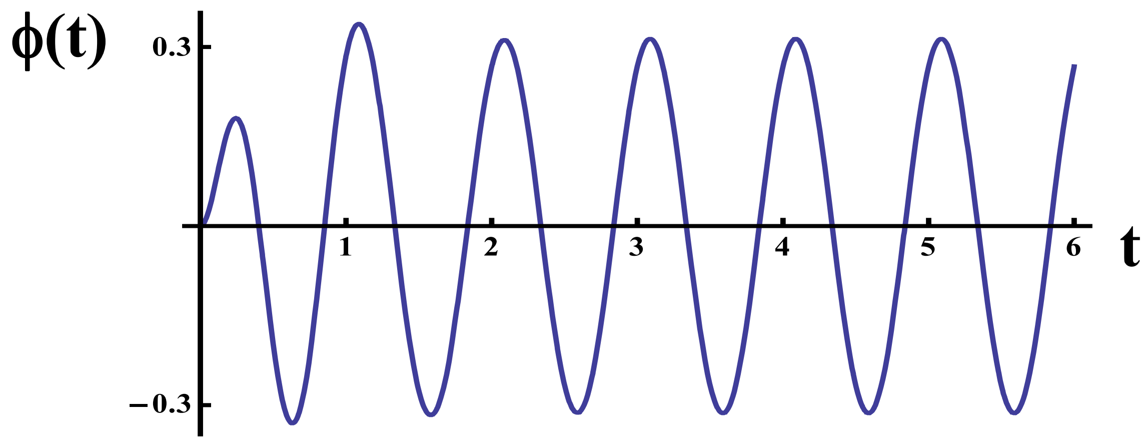
$$\phi(0) = \dot{\phi}(0) = 0$$

Small Oscillations of the Driven, Damped Pendulum

γ << 1 will give small oscillations

Linear $$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \gamma\omega_0^2\cos(\omega t)$$

γ = 0.2



After the initial transient dies out, the solution looks like

$$\phi(t) = A\cos(\omega t - \delta)$$

Periodic "attractor"

Small Oscillations of the Driven, Damped Pendulum

$\gamma << 1$ will give small oscillations

1) **The motion approaches a unique periodic attractor independent of initial conditions**

2) **The motion is sinusoidal with the same frequency as the drive**

$$\phi(t) = A\cos(\omega t - \delta)$$

Moderate Oscillations of the Driven, Damped Pendulum

$\gamma < 1$ and the nonlinearity becomes significant…

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\left(\phi - \frac{1}{6}\phi^3\right) \cong \gamma\omega_0^2 \cos(\omega t)$$

Try $\phi(t) = A\cos(\omega t - \delta)$

This solution gives from the ϕ^3 term: $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

Since there is no cos(3ωt) on the RHS, it must be that $\phi, \dot{\phi}, \ddot{\phi}$ **all develop a cos(3ωt) time dependence. Hence we expect:**

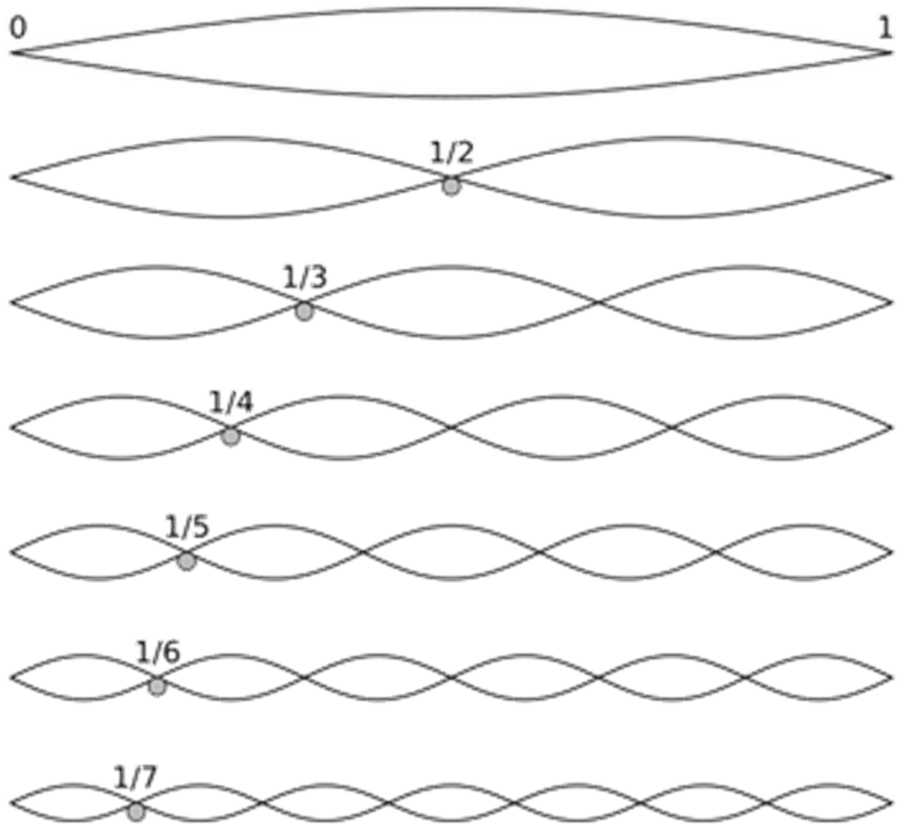
$$\phi(t) = A\cos(\omega t - \delta) + B\cos[3(\omega t - \delta)]$$

$$B << A$$

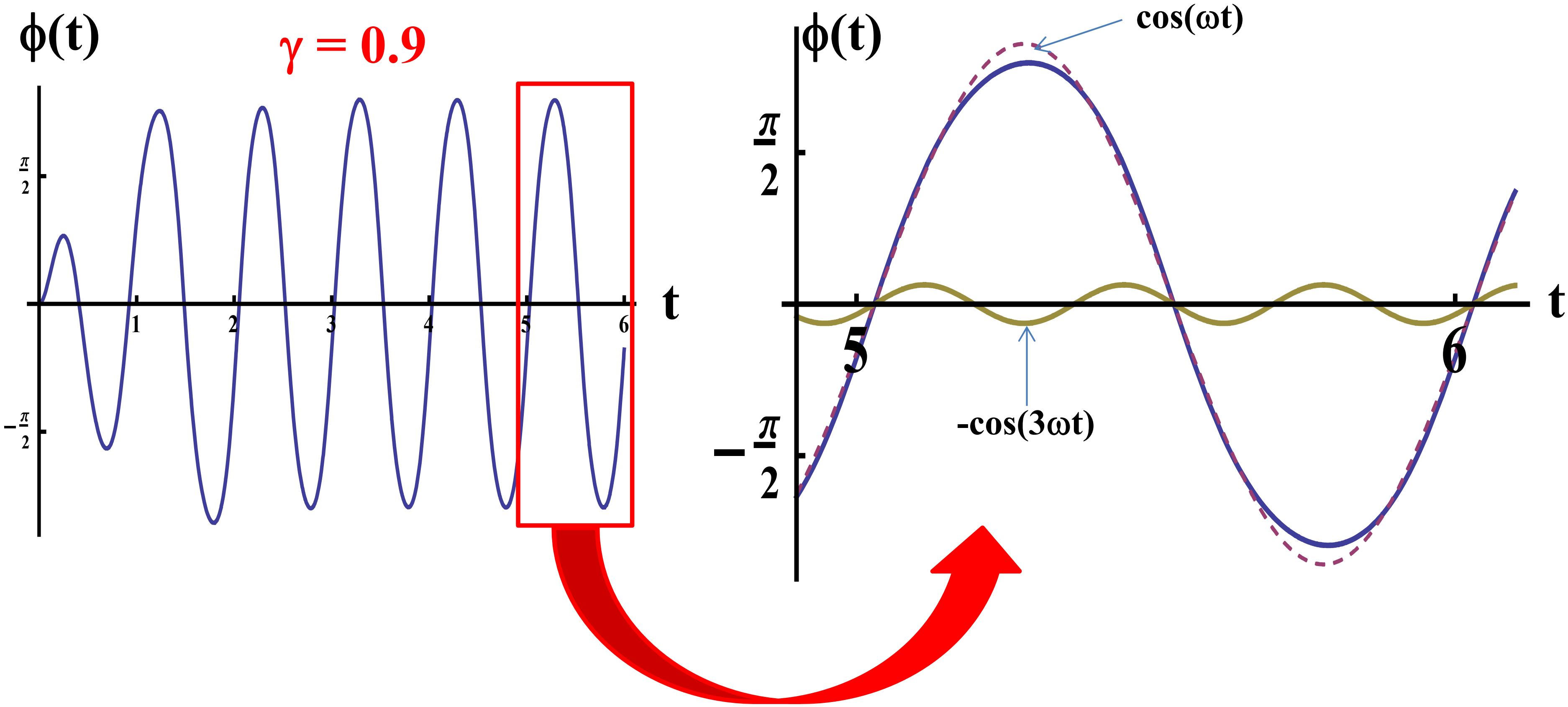
We expect to see a <u>third harmonic</u> as the driving force grows

Harmonics

- Frequency = $n\omega$



Moderate Driving: The Nonlinearity Distorts the cos(ωt)

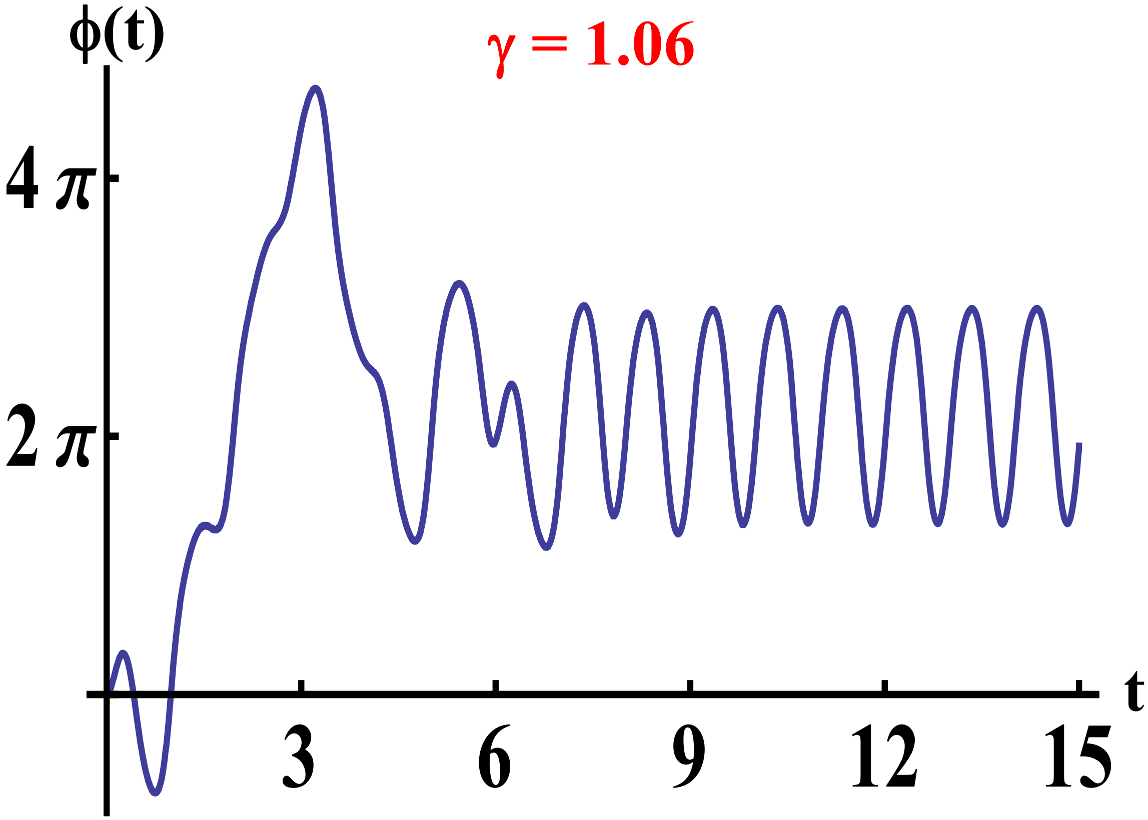


The motion is periodic, but …

The <u>third harmonic</u> distorts the simple $\phi(t) = A\cos(\omega t - \delta)$

Even Stronger Driving: Complicated Transients – then Periodic!

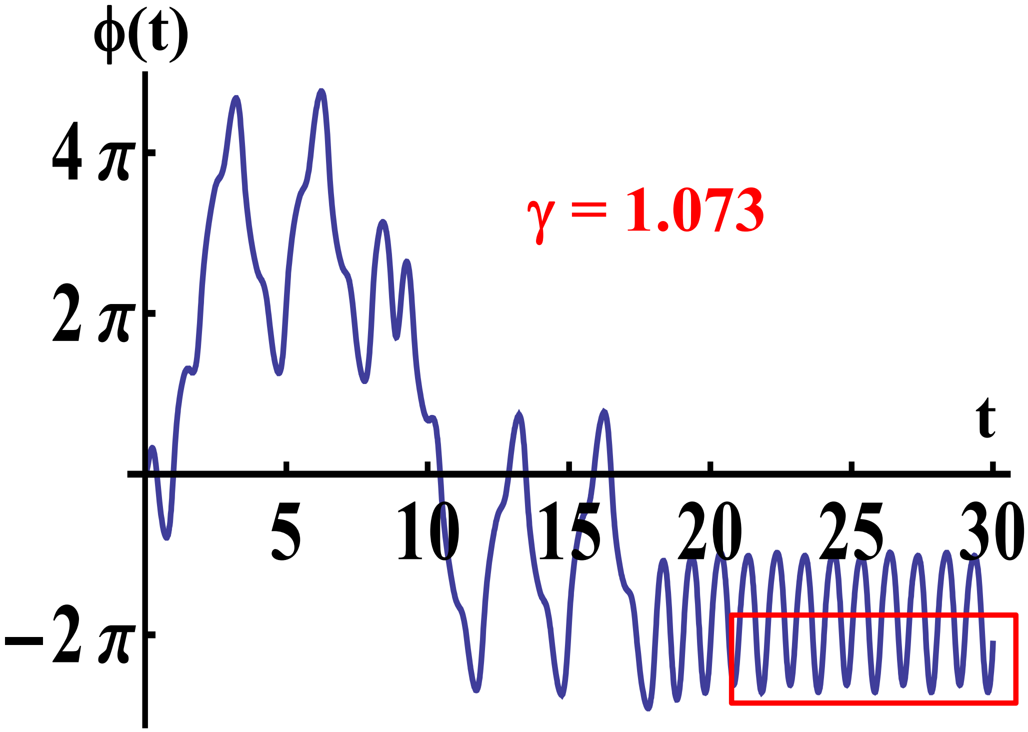
After a wild initial transient, the motion becomes periodic

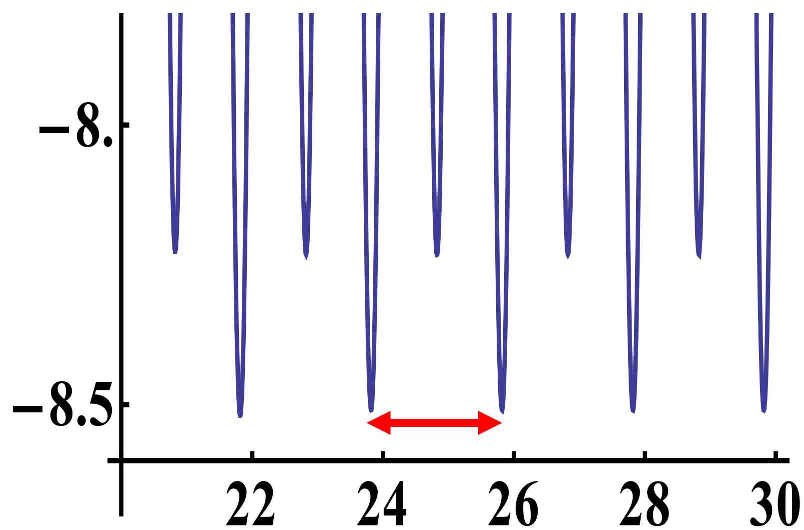


After careful analysis of the long-term motion, it is found to be periodic with the same period as the driving force

Slightly Stronger Driving: Period Doubling

After a wilder initial transient, the motion becomes periodic, but period 2!

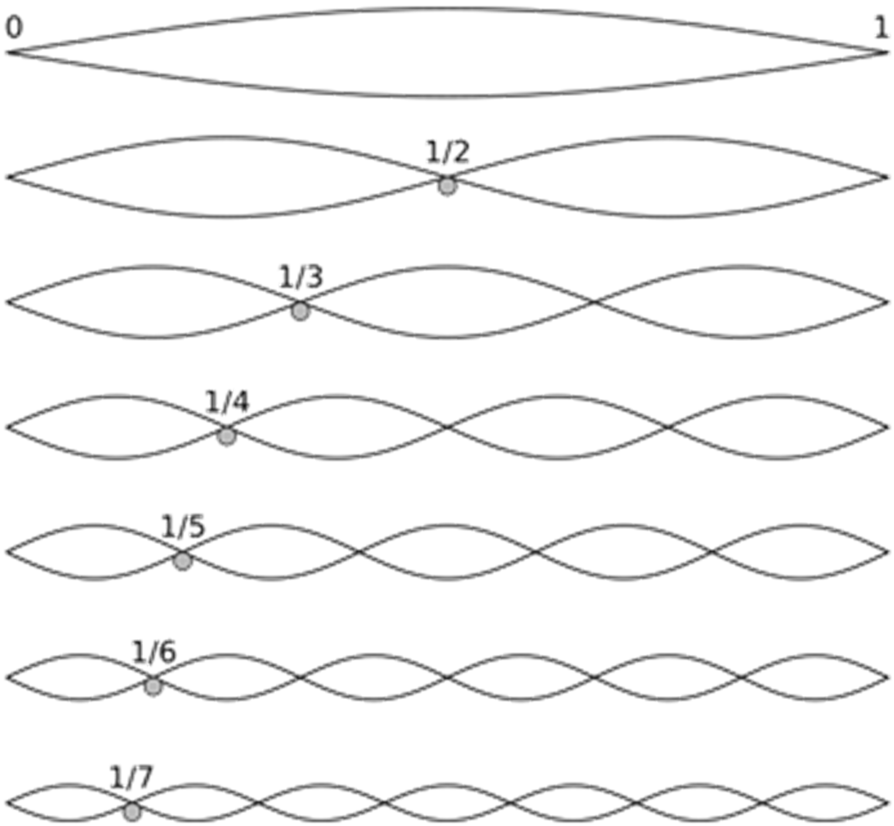




The long-term motion is TWICE the period of the driving force!

A SUB-Harmonic has appeared

Harmonics and Subharmonics



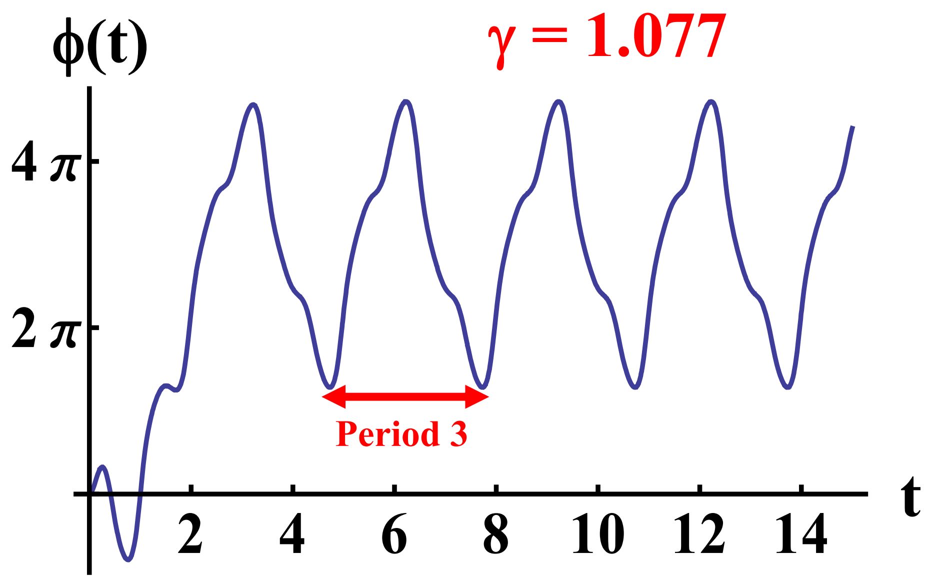
- Frequency = $\frac{\omega}{n}$

- Frequency = $n\omega$

n is an integer

Slightly Stronger Driving: Period 3

The period-2 behavior still has a strong period-1 component
Increase the driving force slightly and we have a very strong period-3 component

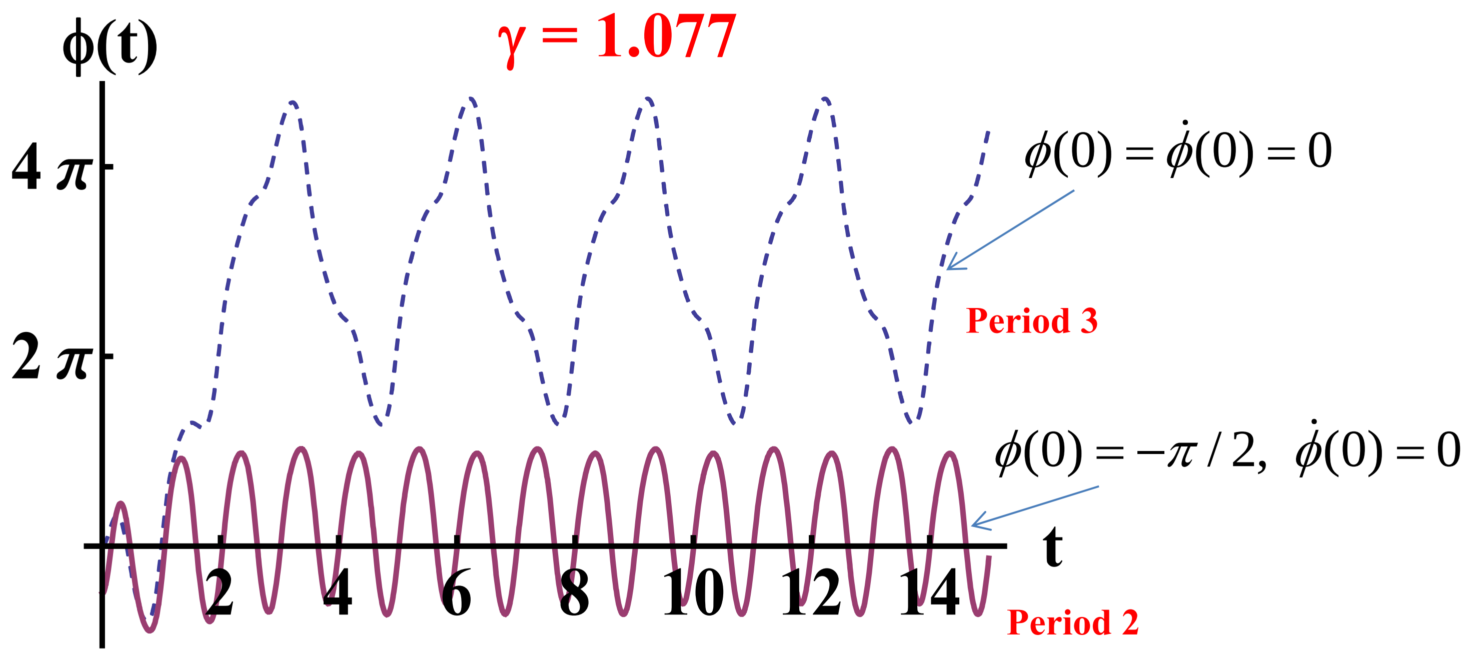


Multiple Attractors

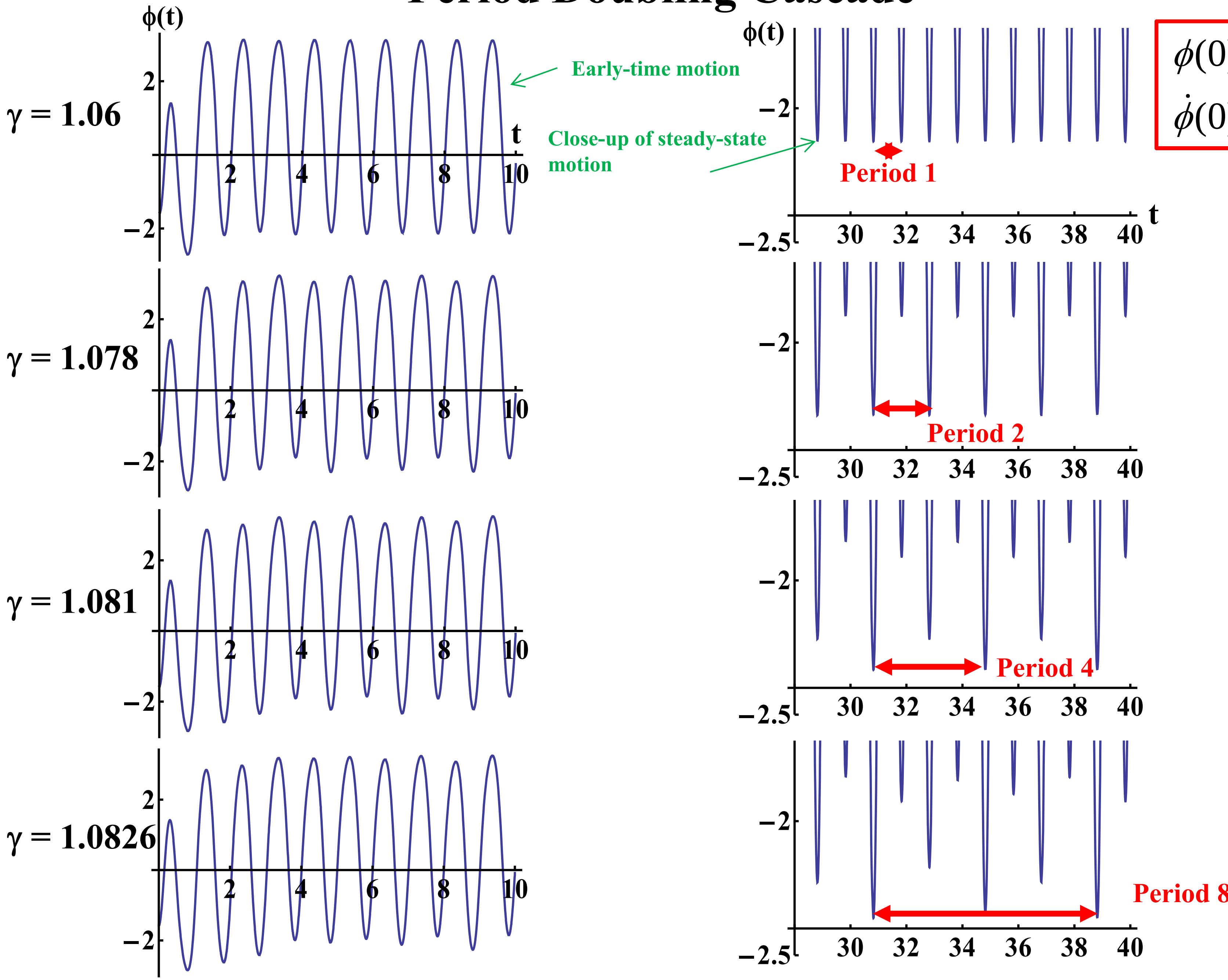
The linear oscillator has a single attractor for a given set of initial conditions

For the drive damped pendulum:

Different initial conditions result in different long-term behavior (attractors)



Period Doubling Cascade

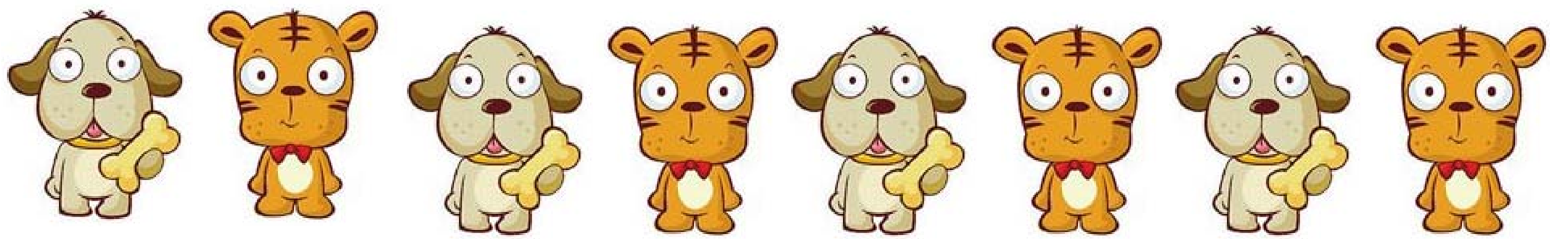


$$\phi(0) = -\pi/2$$
$$\dot{\phi}(0) = 0$$

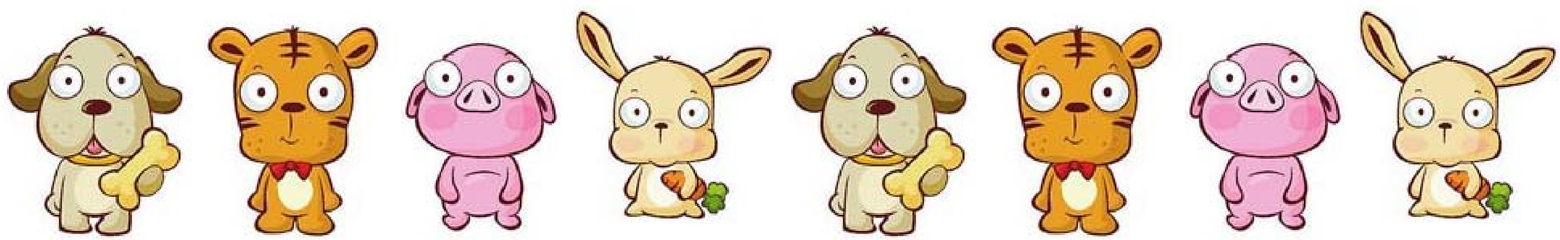
Period Doubling Cascade

γ = 1.06

γ = 1.078



γ = 1.081



γ = 1.0826

‘Bifurcation Points’ in the Period Doubling Cascade

Driven Damped Pendulum

$$\phi(0) = -\pi/2$$

$$\dot{\phi}(0) = 0$$

The spacing between consecutive bifurcation points grows smaller at a steady rate:

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

‘≈’ → ‘=’ as n → ∞

δ = 4.6692016 is called the Feigenbaum number

The limiting value as n → ∞ is γ_c = 1.0829. Beyond that is … chaos!

Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of γ

Such period-doubling cascades are seen in many nonlinear systems
Their form is essentially the same in all systems – it is "universal"

Period infinity

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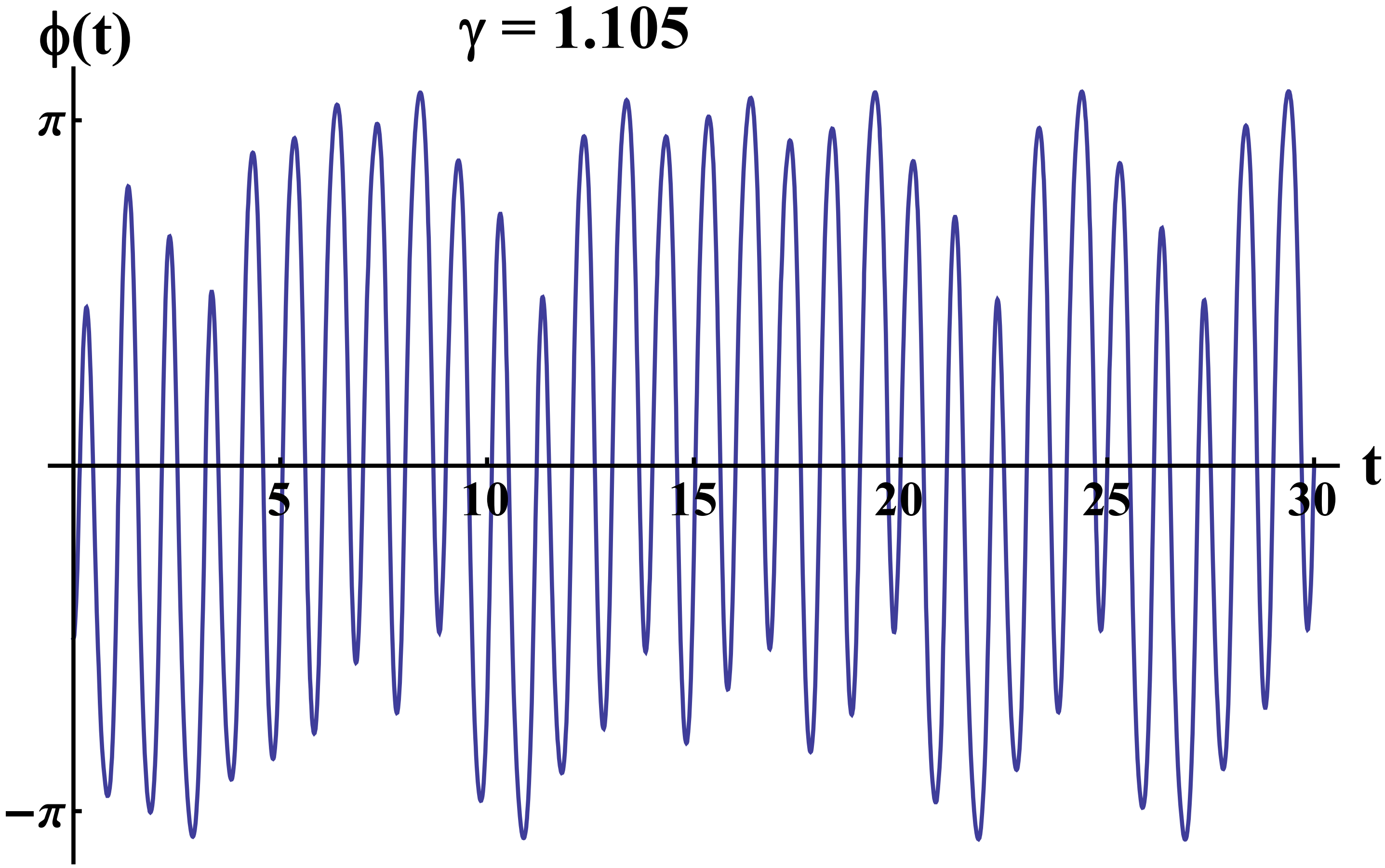
Chaos!



The pendulum is "trying" to oscillate at the driving frequency, but the motion remains erratic for all time

Chaos

- Nonperiodic
- Sensitivity to initial conditions

Sensitivity of the Motion to Initial Conditions

Start the motion of two identical pendulums with slightly different initial conditions

Does their motion converge to the same attractor?

Does it diverge quickly?

Two pendulums $\phi_1(t), \quad \phi_2(t)$ **are given different initial conditions**

Follow their evolution and calculate $\Delta\phi(t) = \phi_2(t) - \phi_1(t)$

For a <u>linear</u> oscillator $\phi(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Long-term attractor

Transient behavior

$r_{1,2} = -\beta \pm i\omega_1$ **underdamping,** $\omega_1 = \sqrt{\omega^2 - \beta^2}$

The initial conditions affect the transient behavior, the long-term attractor is the same

Hence $\Delta\phi(t) = De^{-\beta t}\cos(\omega_1 t - \delta_1)$

Thus the trajectories will converge after the transients die out

Convergence of Trajectories in Linear Motion

$$\Delta\phi(t) = De^{-\beta t}\cos(\omega_1 t - \delta_1)$$

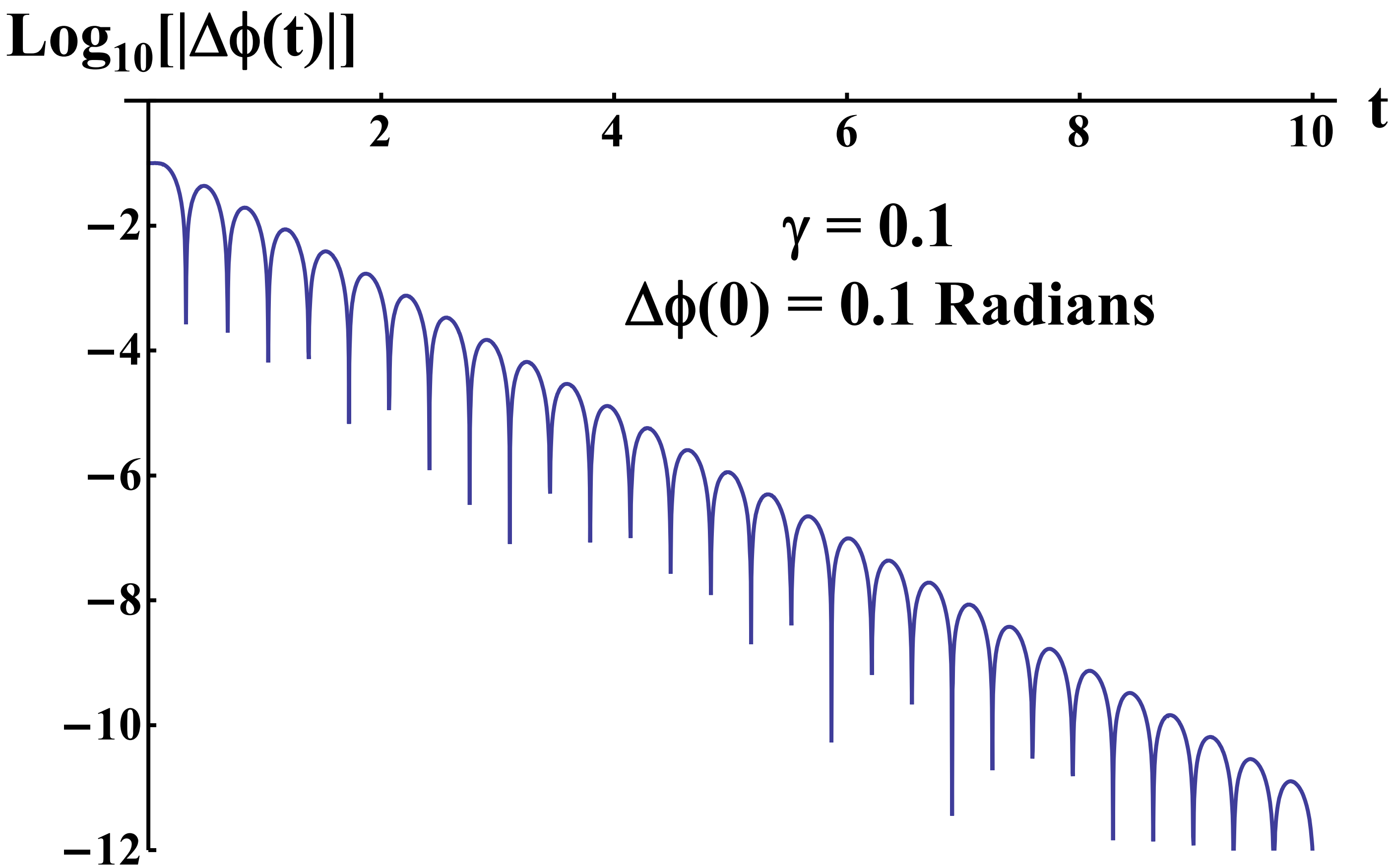
Take the logarithm of $|\Delta\phi(t)|$ to magnify small differences.

$$\ln[|\Delta\phi(t)|] = \ln(D) - \beta t + \ln\left[|\cos(\omega_1 t - \delta_1)|\right]$$

Plotting $\log_{10}[|\Delta\phi(t)|]$ vs. t should be a straight line of slope $-\beta$, plus some wiggles from the $\ln[|\cos(\omega_1 t - \delta_1)|]$ term

Note that $\log_{10}[x] = \log_{10}[e] \ln[x]$

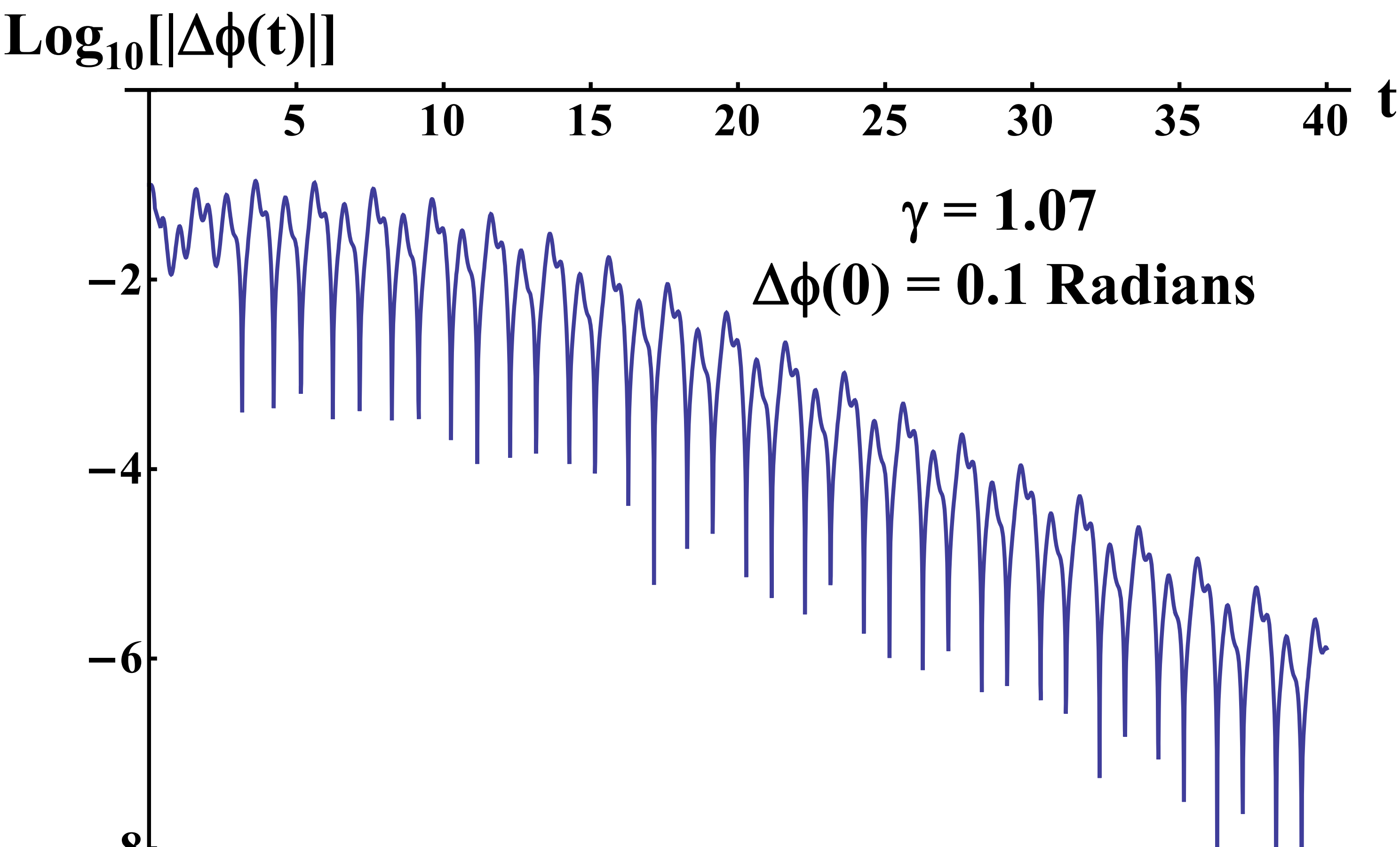
Convergence of Trajectories in Linear Motion



The trajectories converge quickly for the small driving force (~ linear) case

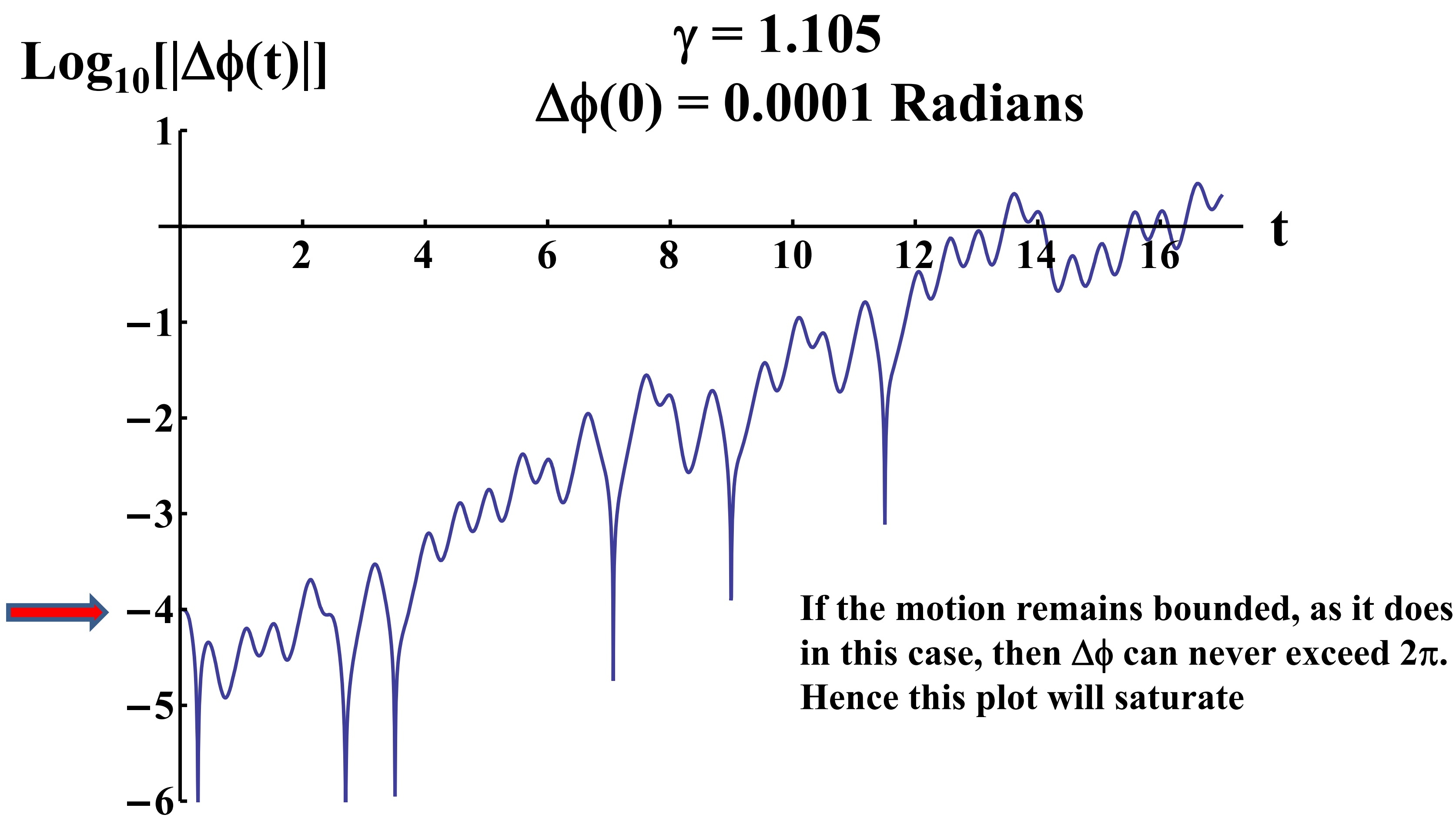
This shows that the linear oscillator is essentially insensitive to its initial conditions!

Convergence of Trajectories in Period-2 Motion



The trajectories converge more slowly, but still converge

<u>Divergence</u> of Trajectories in <u>Chaotic</u> Motion



The trajectories diverge, even when very close initially

$\Delta\phi(16) \sim \pi$, so there is essentially complete loss of correlation between the pendulums

Extreme Sensitivity to Initial Conditions

Practically impossible to predict the motion

The Lyapunov Exponent

$$\left|\Delta\phi(t)\right| \sim Ke^{\lambda t} \qquad K > 0$$

λ = Lyapunov exponent

λ < 0: periodic motion in the long term

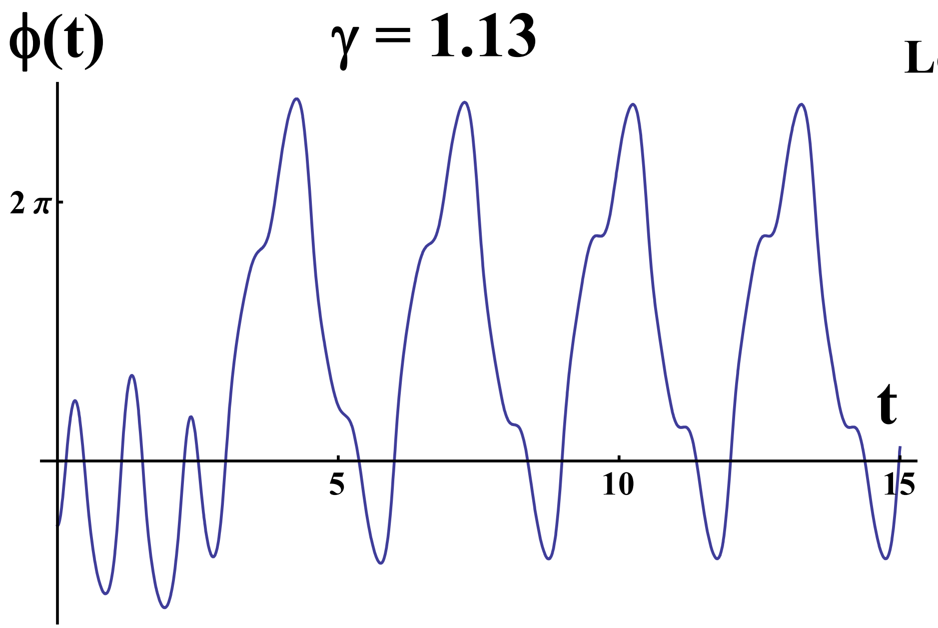
λ > 0: chaotic motion

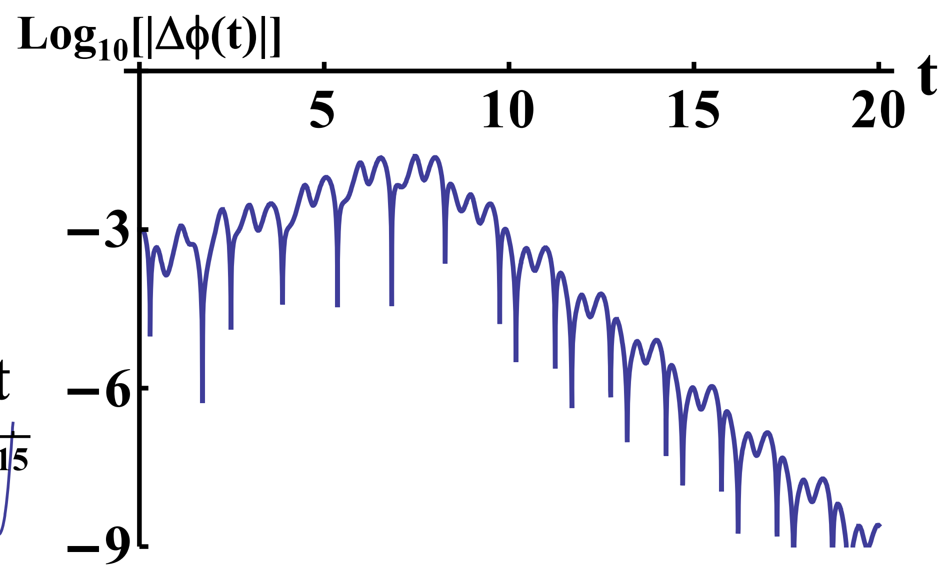
Linear	Nonlinear	
		Chaos
Drive period	Harmonics, Subharmonics, Period-doubling	Nonperiodic, Extreme sensitivity
$\lambda < 0$	$\lambda < 0$	$\lambda > 0$

What Happens if we Increase the Driving Force Further?

Does the chaos become more intense?

$\Delta\phi(0) = 0.001$ Radians





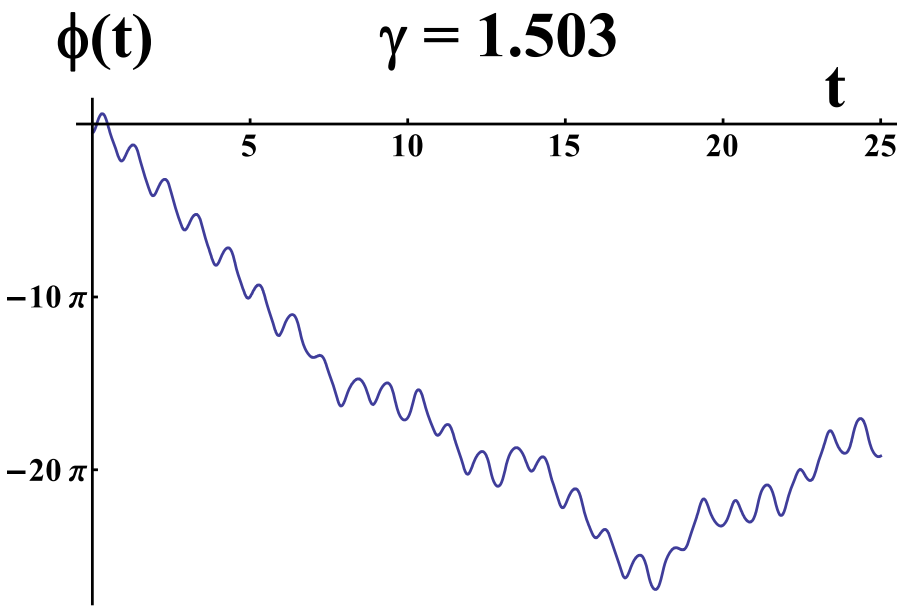
Period 3 motion re-appears!

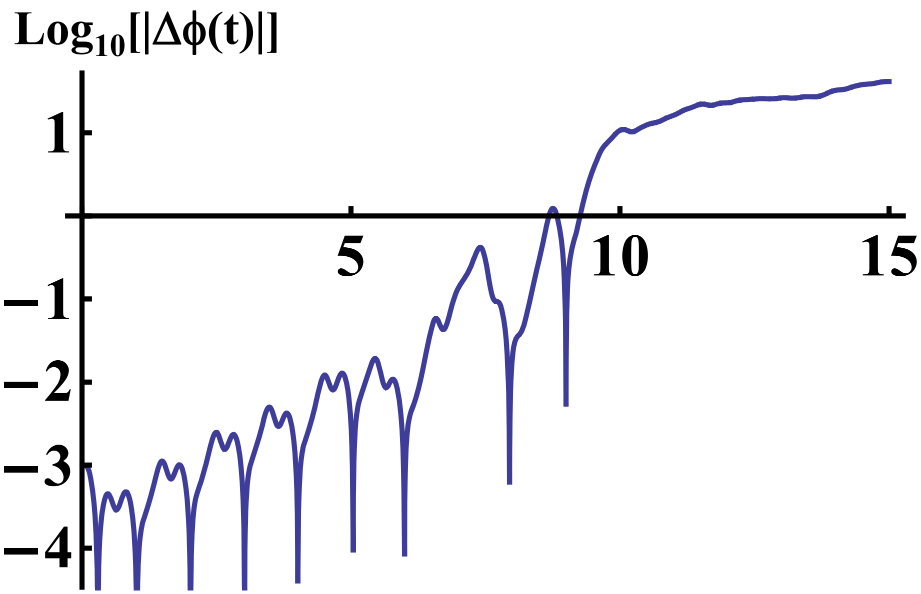
With increasing γ the motion alternates between chaotic and periodic

What Happens if we Increase the Driving Force Further?

Does the chaos re-appear?

$\Delta\phi(0) = 0.001$ Radians

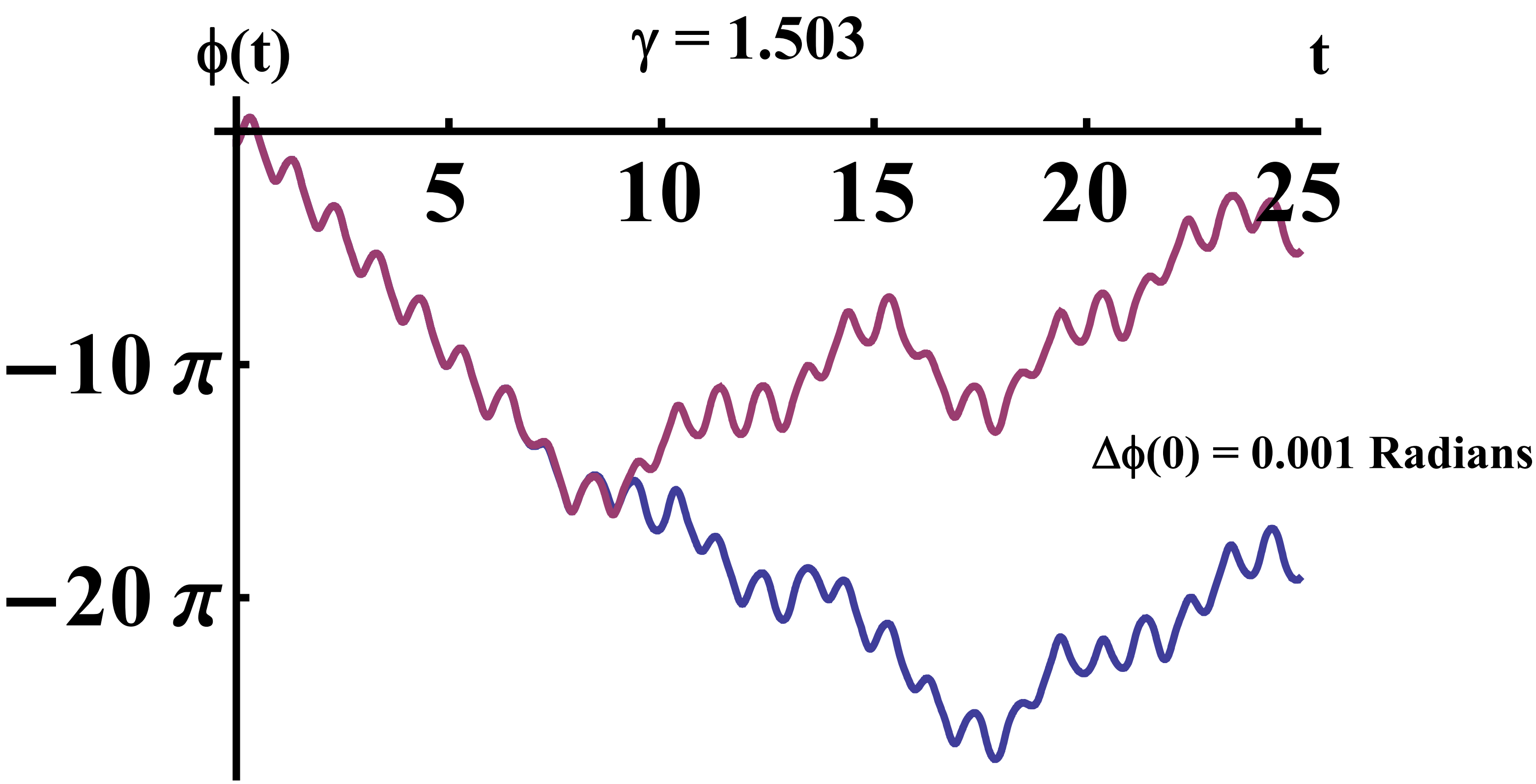




Chaotic motion re-appears!

This is a kind of ‘rolling’ chaotic motion

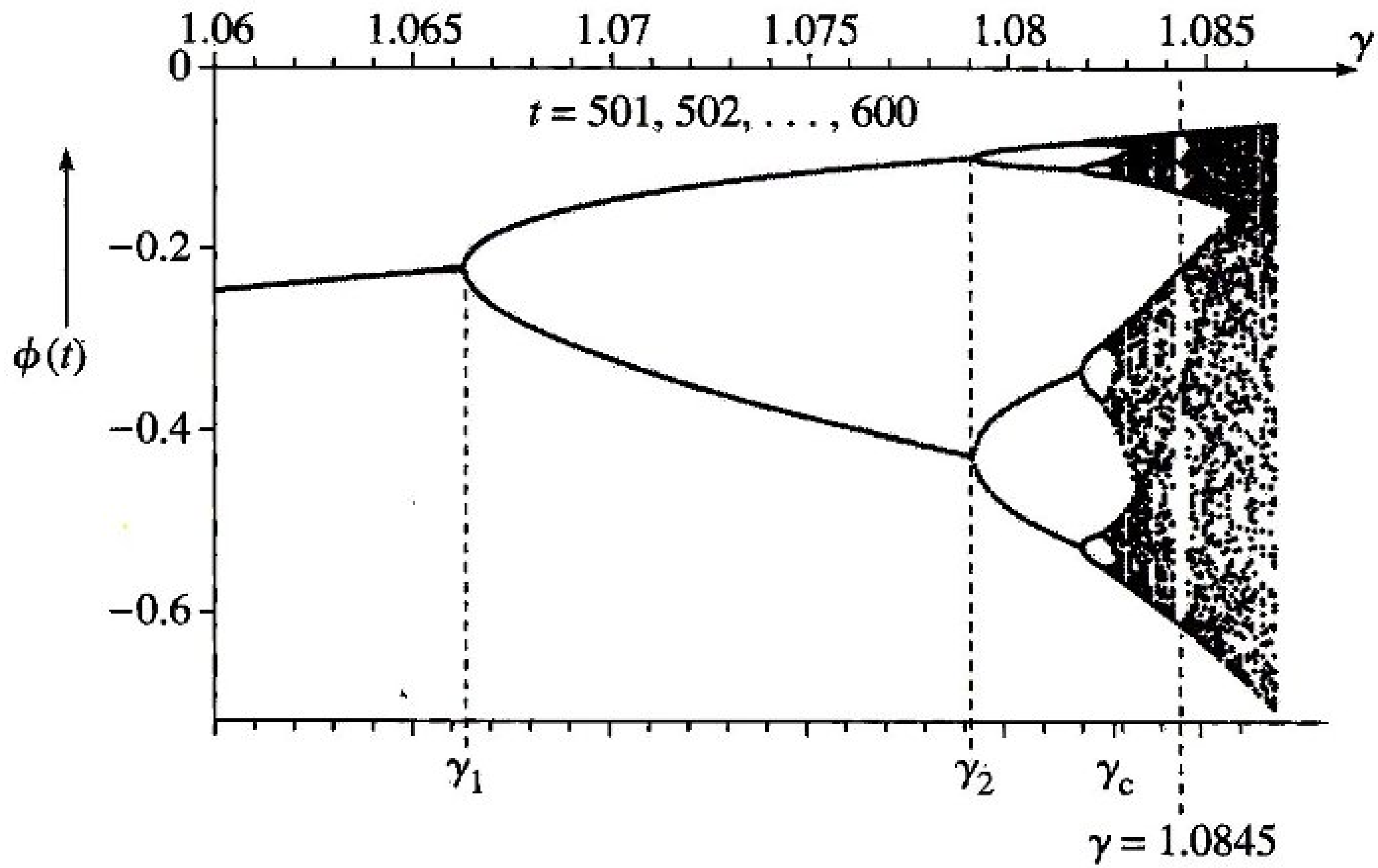
Divergence of Two Nearby Initial Conditions for Rolling Chaotic Motion



Chaotic motion is always associated with extreme sensitivity to initial conditions

Periodic and chaotic motion occur in narrow intervals of γ

Bifurcation Diagram



Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of γ

Such period-doubling cascades are seen in many nonlinear systems
Their form is essentially the same in all systems – it is "universal"

Driven Diode experiment

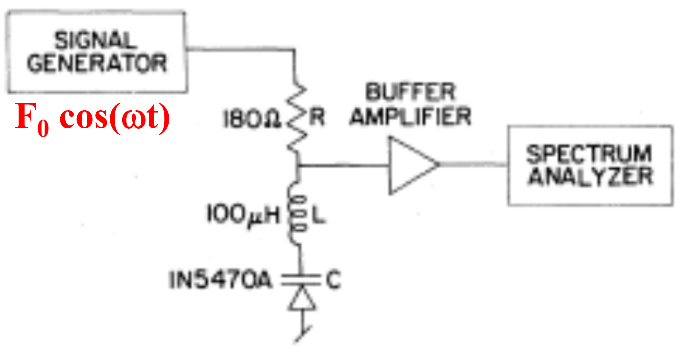


FIG. 1. Experimental apparatus for subharmonic generation.

Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator
Paul S. Linsay

Sub-harmonic frequency spectrum

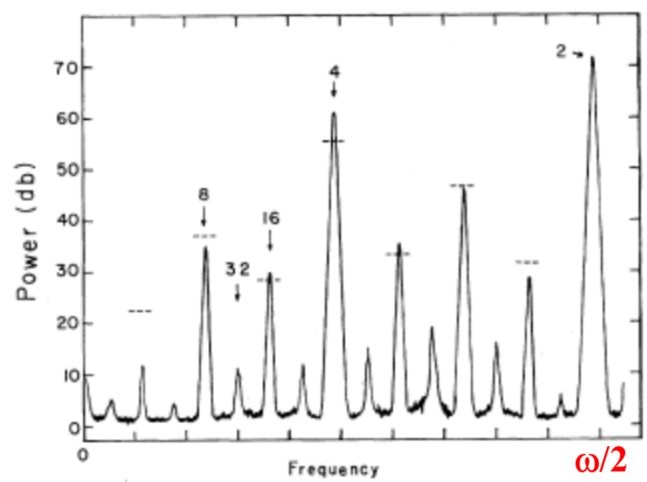


FIG. 5. Power spectral density (dB) vs frequency for f = 98 kHz, dynamic range 70 dB, showing subharmonics to $f/32$. The components agree with prediction (dashed bars, Ref. 14) within 2 dB rms deviation, except for the peak at $f/16$.

Evidence for Universal Chaotic Behavior of a Driven Nonlinear Oscillator
James Testa, José Pérez, and Carson Jeffries

Period Doubling Cascade

Period doubling continues in a sequence of ever-closer values of γ

Such period-doubling cascades are seen in many nonlinear systems
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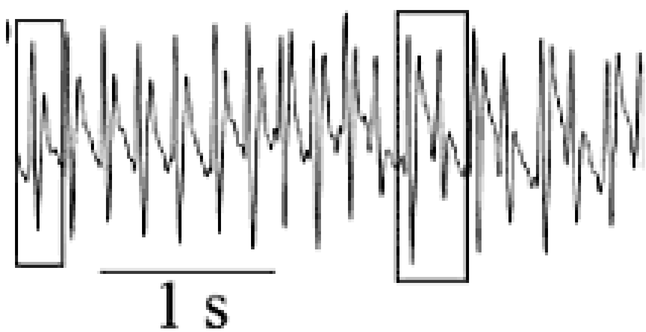
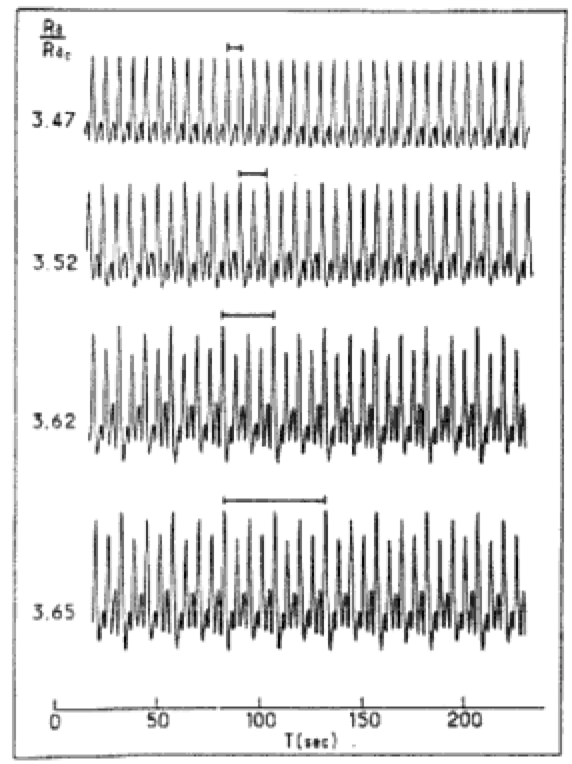


Figure 3.7. Period doubling in seizures and in thermoconvection. (A), intracranial recording from a patient suffering temporal lobe seizures. The left-hand side recordings show how the length of the period, marked by the rectangles, doubles during the seizure, while the right-hand side recording corresponds to a posterior time toward the end of the seizure where more complex aperiodic activity is apparent due to successive period doublings. (B), to illustrate how period doubling was defined in classic studies and the similarity with the intracerebral recording, doubling of the periods in a thermoconvection experiment is shown. The time series is the temperature of a fluid, note the subharmonic cascade as the control parameter of the system changes. The line segments indicate the duration of one period. Panel A is reprinted with permission from Perez Velazquez *et al.* (2003) and panel B from Bergé *et al.* (1984).



A period doubling cascade in convection of mercury in a small convection cell. The plots show the temperature at one fixed point in the cell as a function of time, for four successively larger temperature gradients as given by the parameter R/R_c

The Brain-behaviour Continuum: The Subtle Transition Between Sanity and Insanity
By Jose Luis. Perez Velazquez

Fig. 12.9, Taylor

Bifurcation Diagram

Used to visualize the behavior as a function of driving amplitude γ

1) **Choose a value of γ**
2) **Solve for φ(t), and plot a <u>periodic sampling</u> of the function**

   $\phi(t_0), \quad \phi(t_0+1), \quad \phi(t_0+2), \quad \phi(t_0+3), \quad \phi(t_0+4), \ldots$

   t_0 chosen at a time after the attractor behavior has been achieved
3) **Move on to the next value of γ**

$\phi(0) = -\pi/2$

$\dot{\phi}(0) = 0$

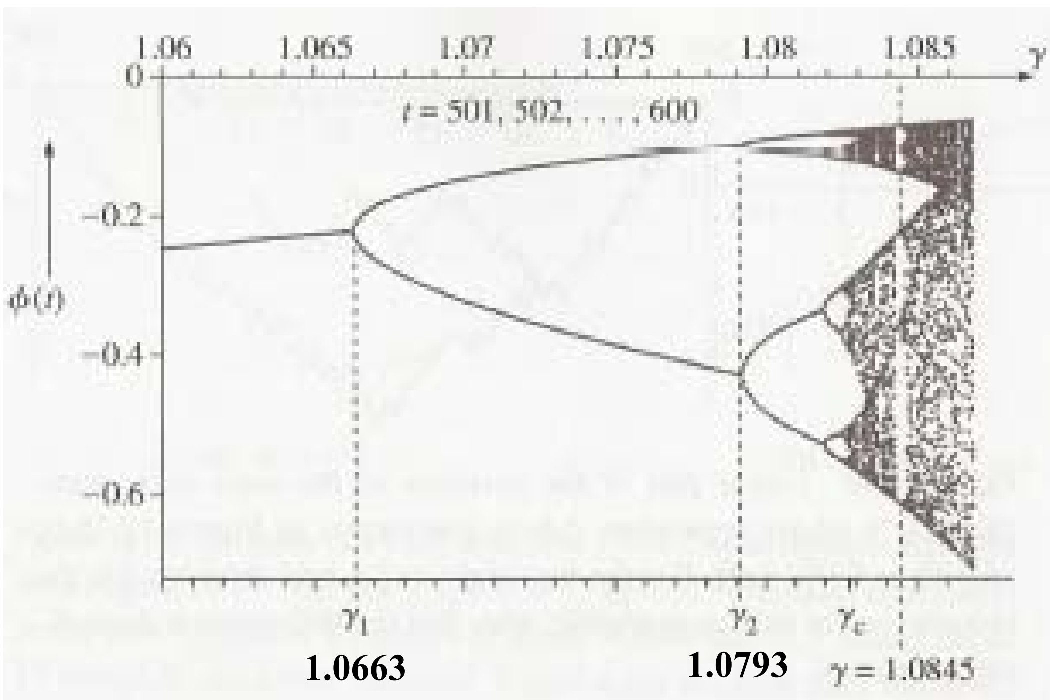
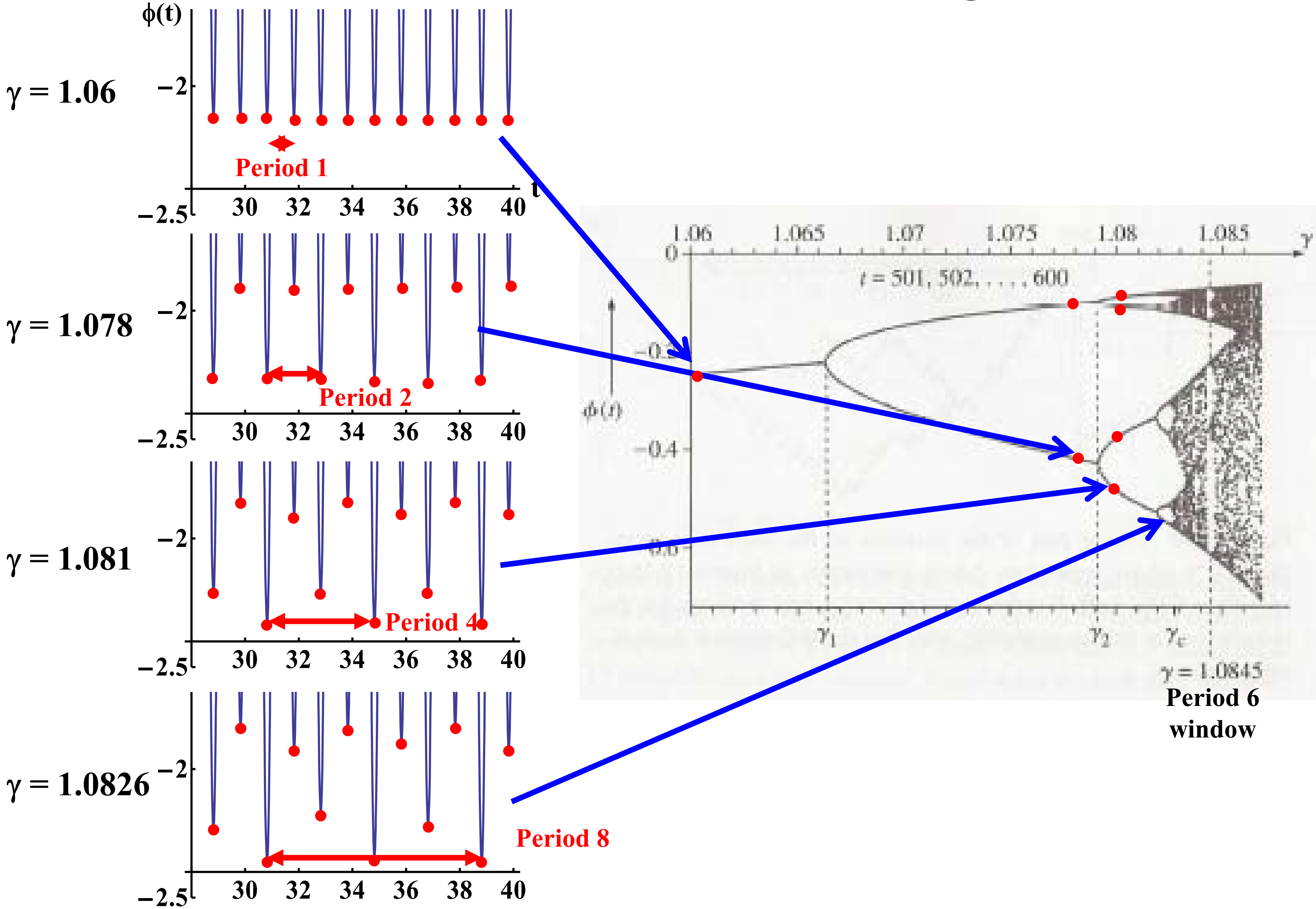
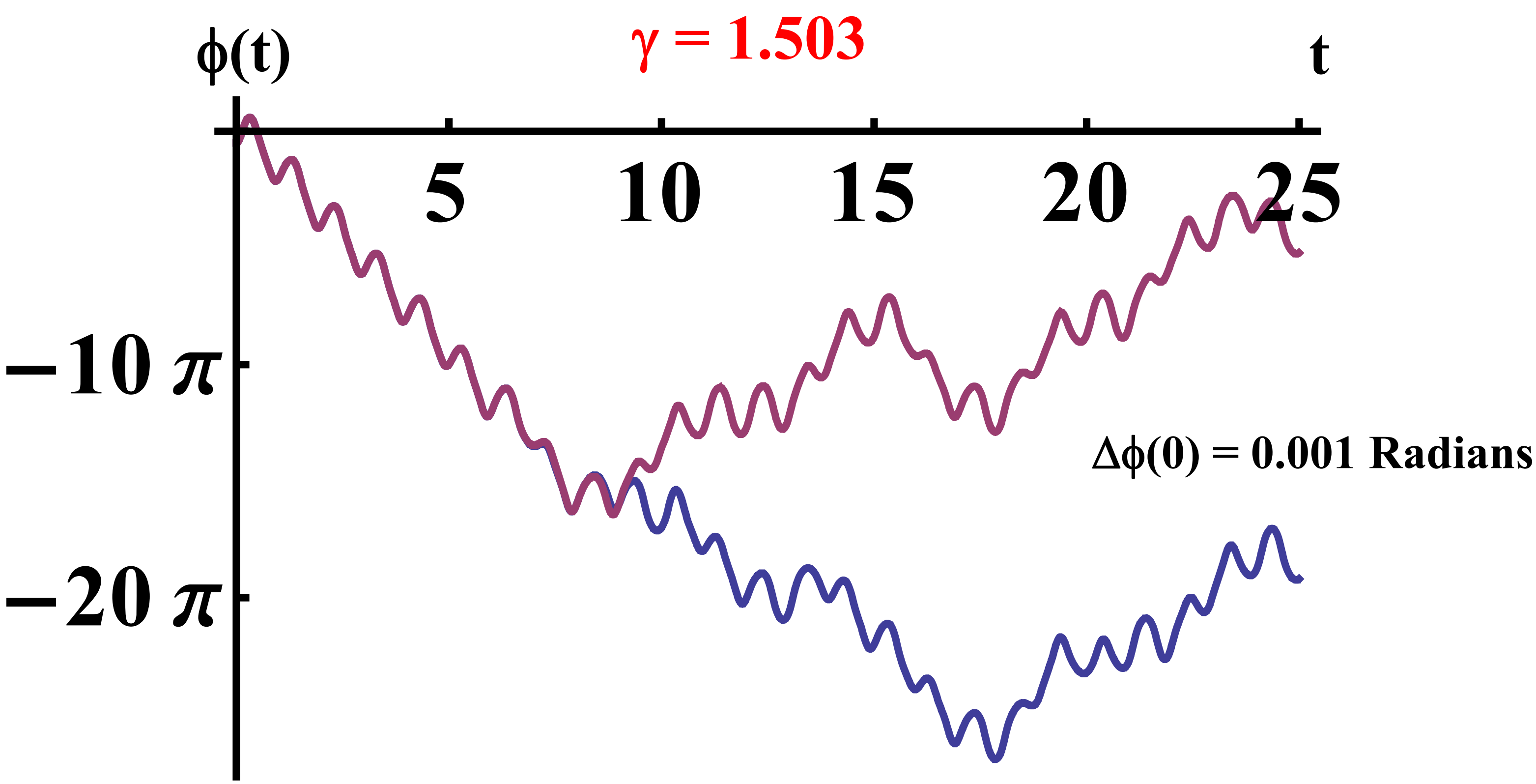


Fig. 12.17

Construction of the Bifurcation Diagram

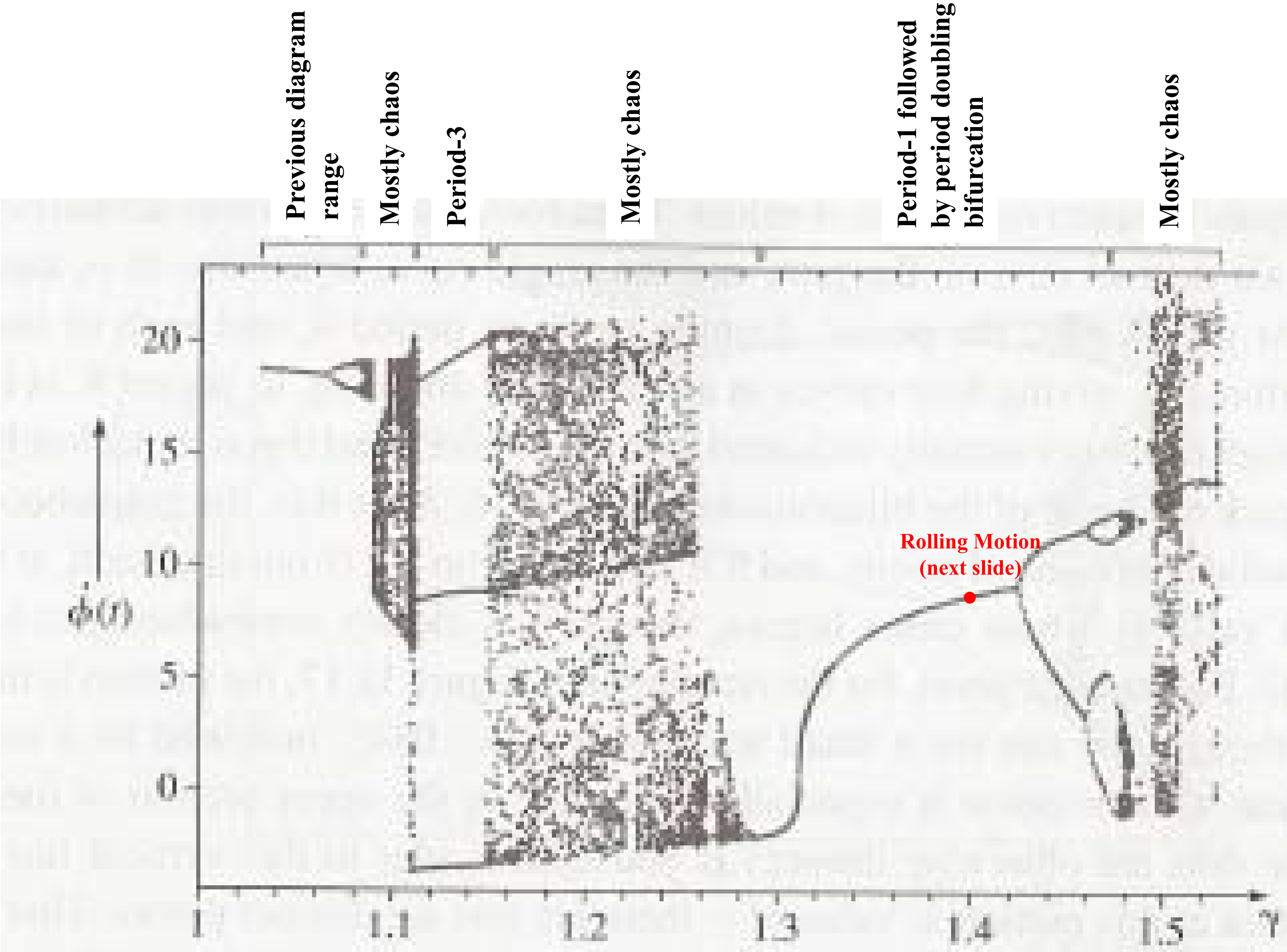


The Rolling Motion Renders the Bifurcation Diagram Useless



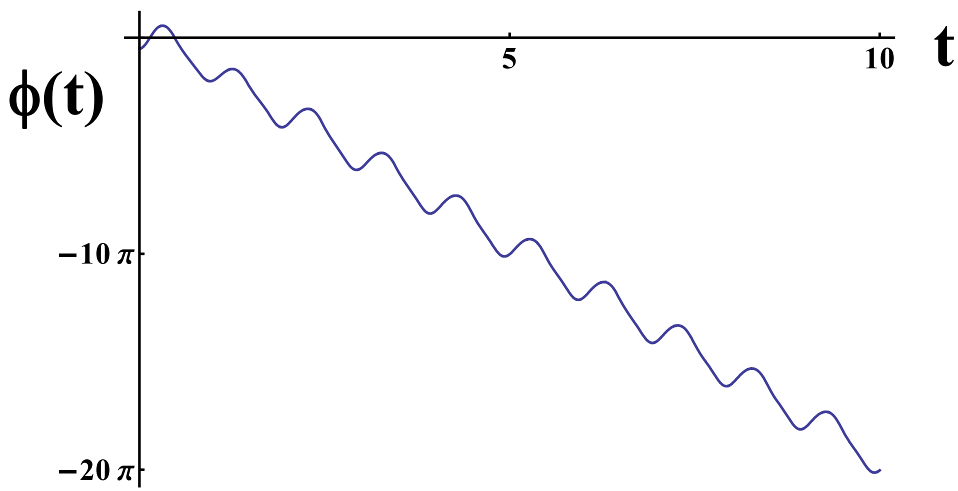
As an alternative, plot $\dot{\phi}(t)$

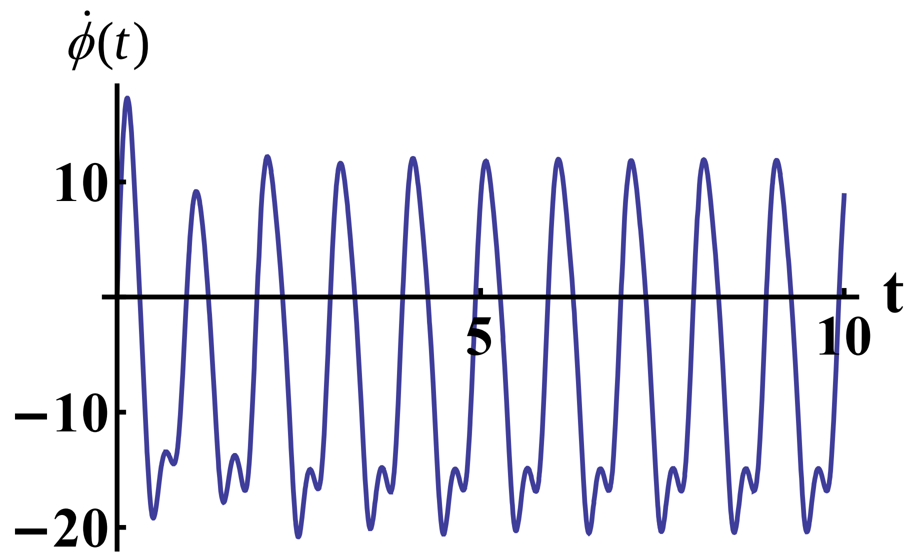
$\dot{\phi}(t)$ Bifurcation Diagram Over a Broad Range of γ



Period-1 Rolling Motion at $\gamma = 1.4$

$\gamma = 1.4$





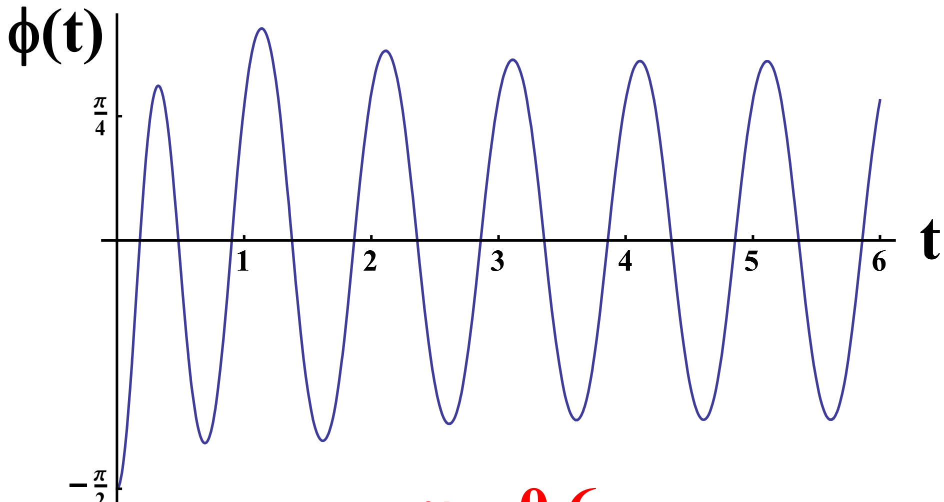
Even though the pendulum is "rolling", $\dot{\phi}(t)$ is periodic

An Alternative View: State Space Trajectory

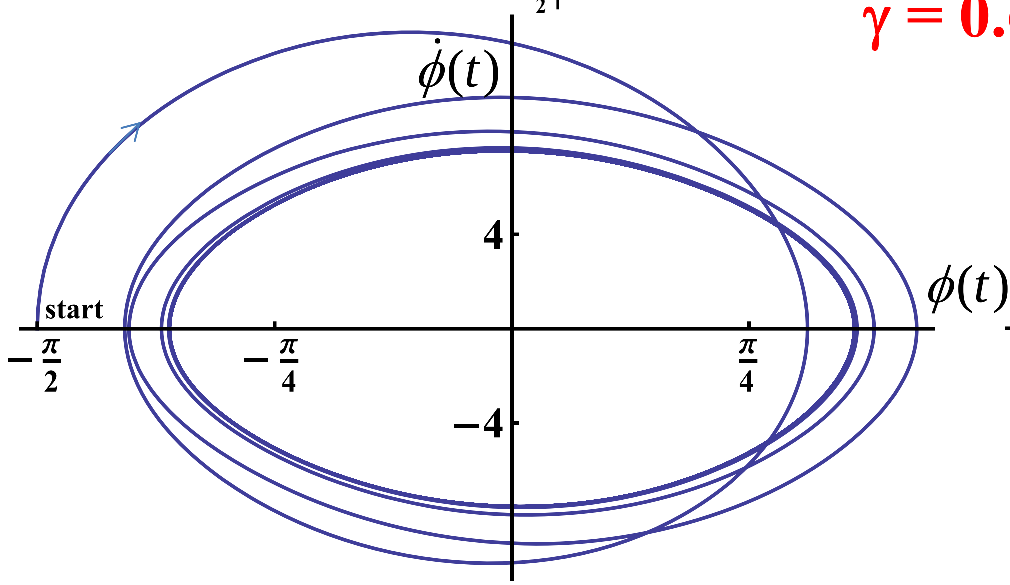
Plot $\dot{\phi}(t)$ vs. $\phi(t)$ with time as a parameter

$\phi(0) = -\pi/2$

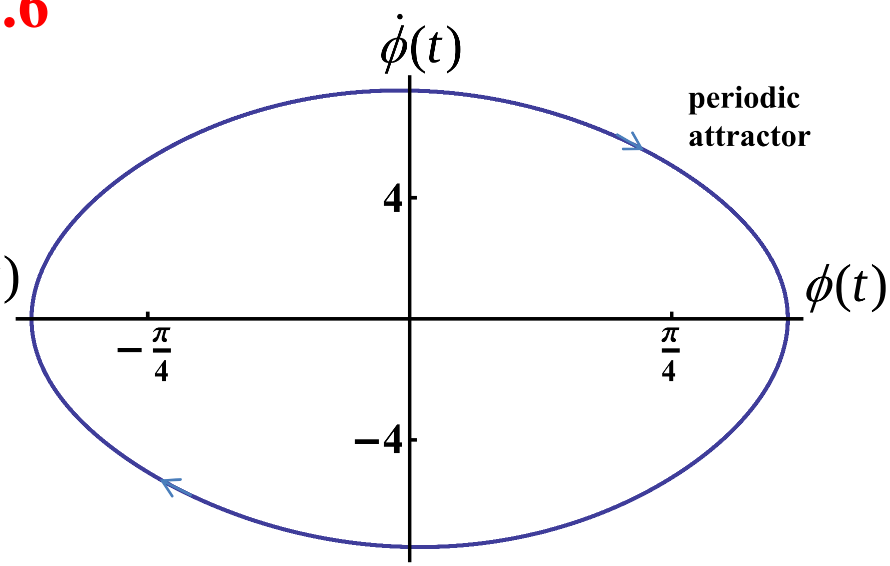
$\dot{\phi}(0) = 0$



γ = 0.6



First 20 cycles



Cycles 5 -20

Fig. 12.20, 12.21

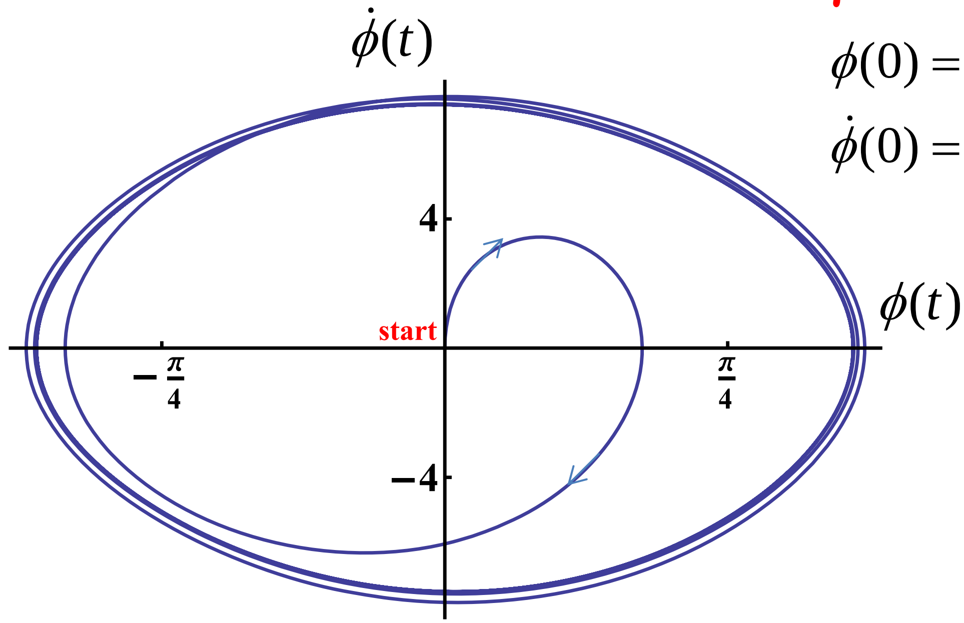
An Alternative View: State Space Trajectory

Plot $\dot{\phi}(t)$ vs. $\phi(t)$ with time as a parameter

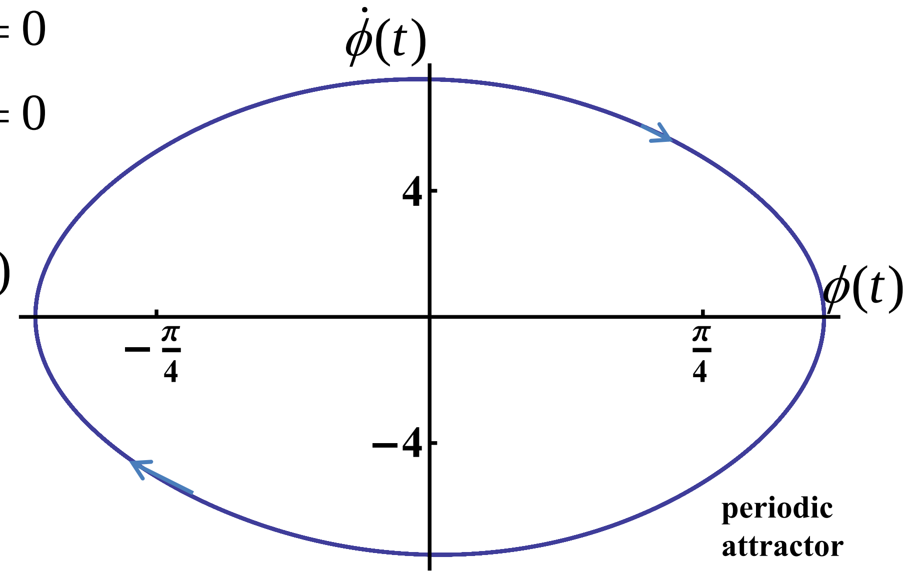
$\gamma = 0.6$

$\phi(0) = 0$

$\dot{\phi}(0) = 0$



First 20 cycles



Cycles 5 -20

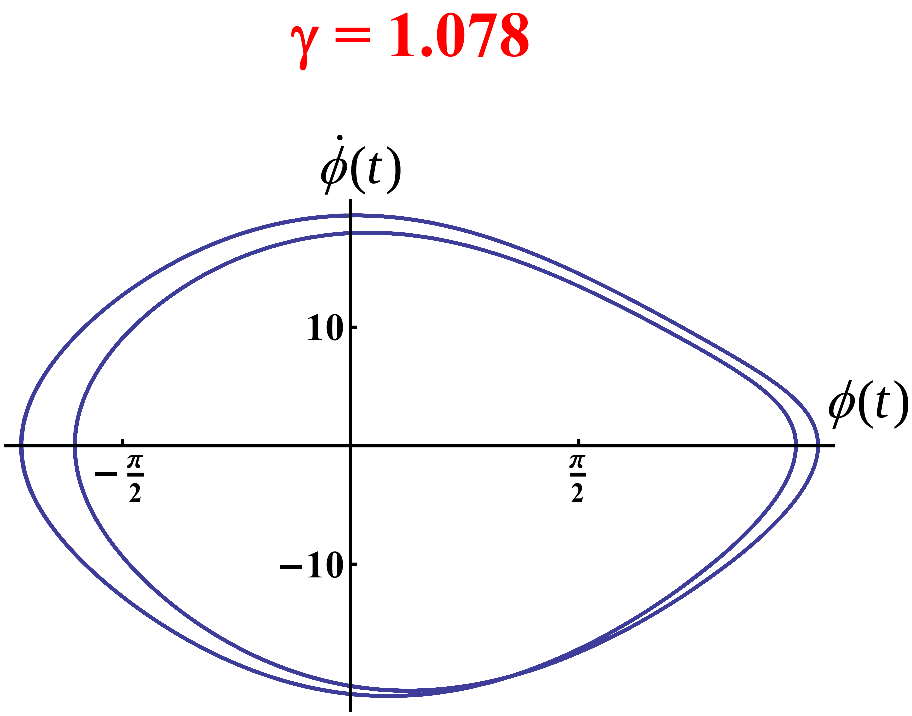
The periodic attractor:
$[\phi(t),\dot{\phi}(t)]$ is an ellipse

$$\phi(t) = A\cos(\omega t - \delta)$$
$$\dot{\phi}(t) = -A\omega\sin(\omega t - \delta)$$

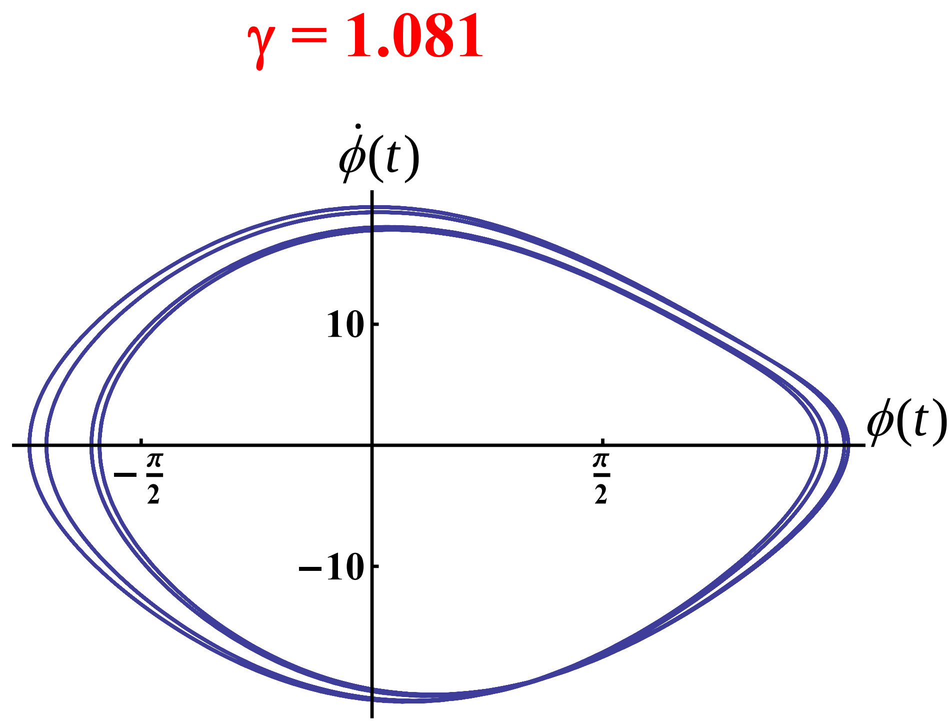
The state space point moves clockwise on the orbit

Fig. 12.22

State Space Trajectory for Period Doubling Cascade



Period-2



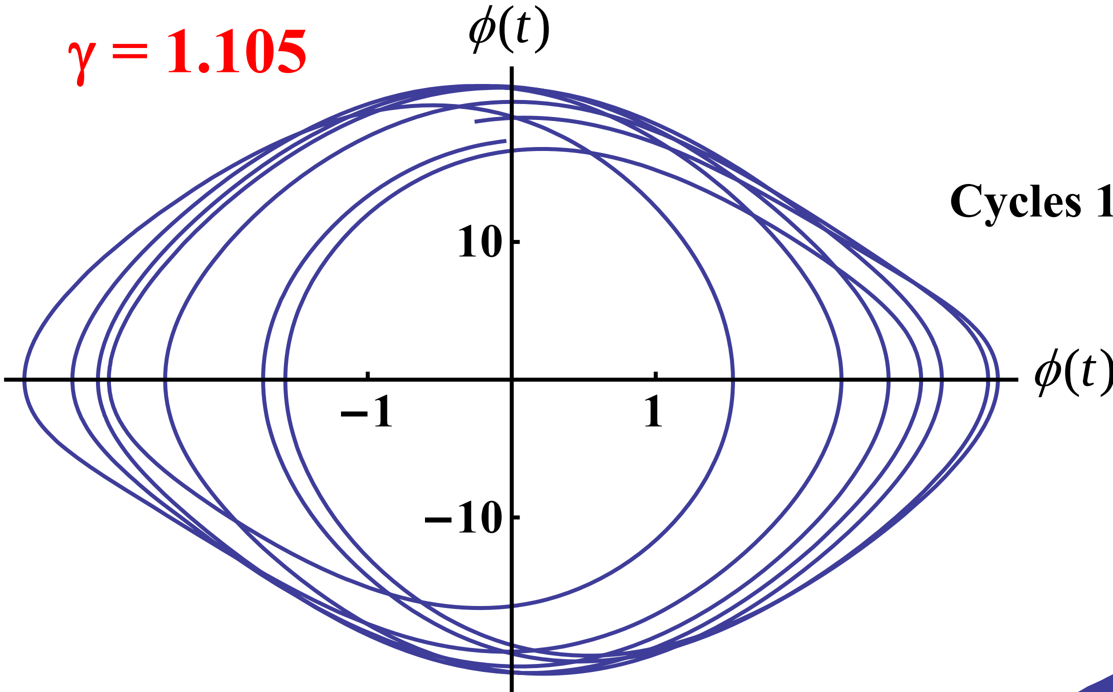
Period-4

Plotting cycles 20 to 60

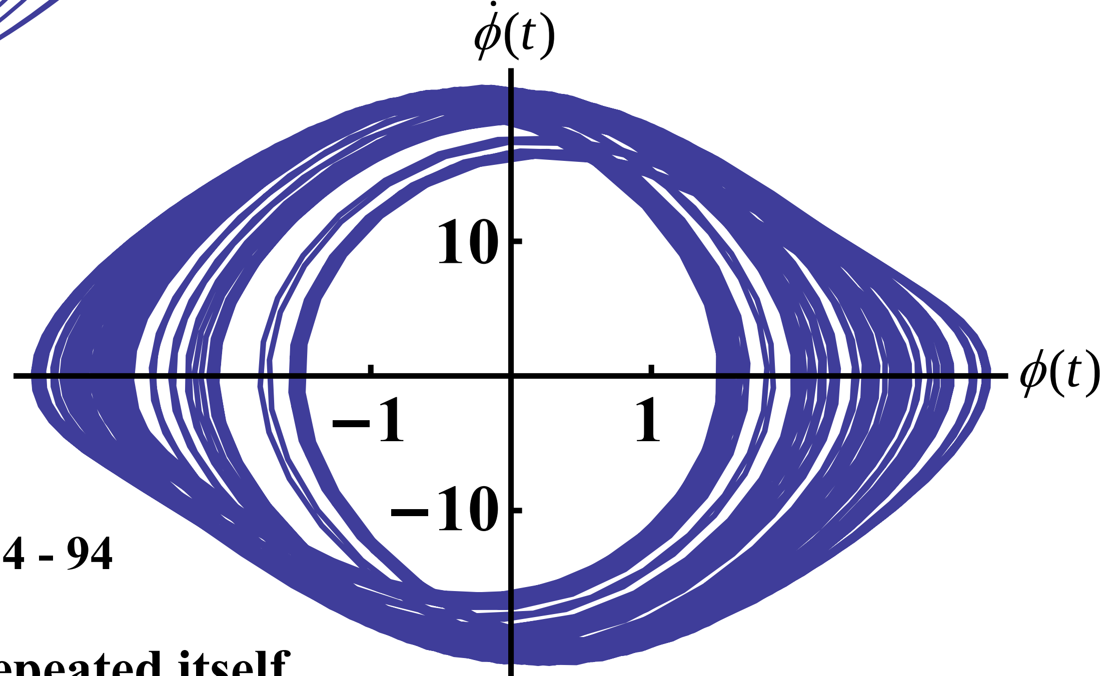
Fig. 12.23

State Space Trajectory for Chaos

$\gamma = 1.105$



Cycles 14 - 21



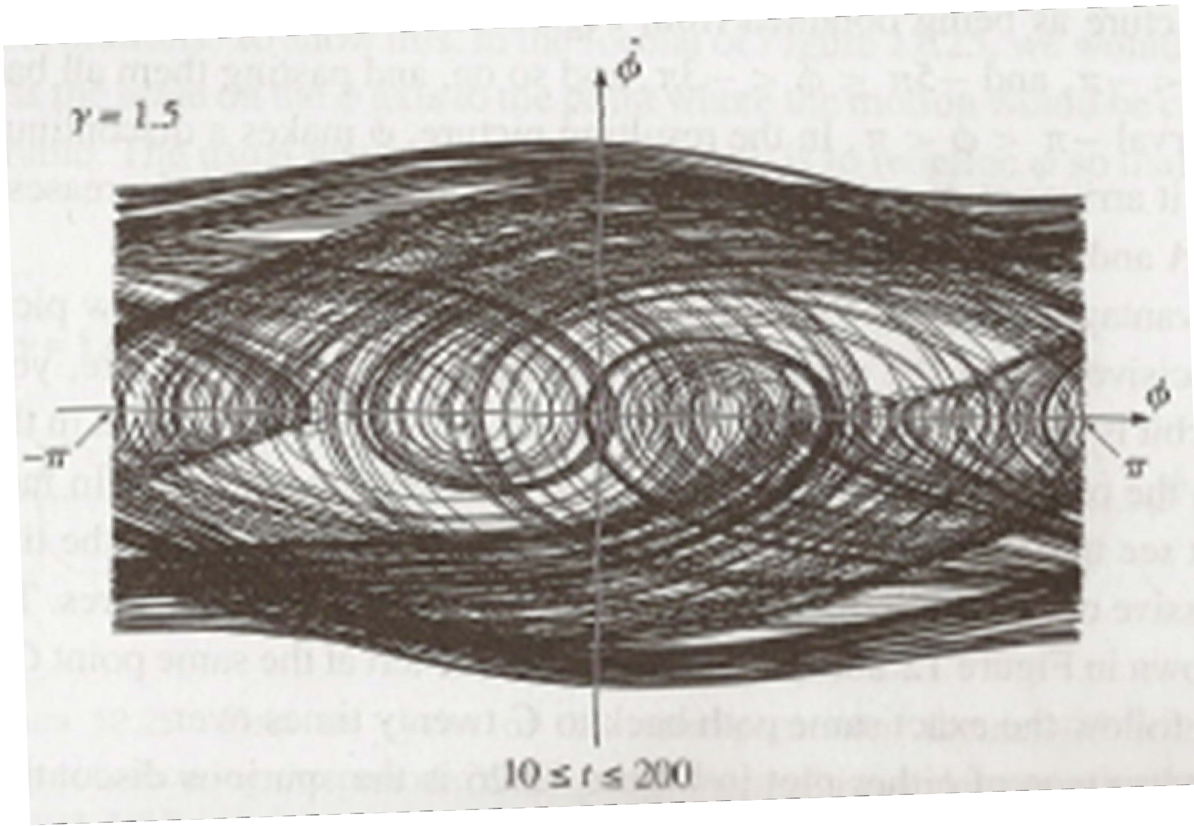
Cycles 14 - 94

The orbit has not repeated itself…

State Space Trajectory for Chaos

$\gamma = 1.5$
$\beta = \omega_0/8$

Chaotic rolling motion
Mapped into the interval $-\pi < \phi < \pi$



Cycles 10 – 200

This plot is still quite messy. There's got to be a better way to visualize the motion …

The Poincaré Section

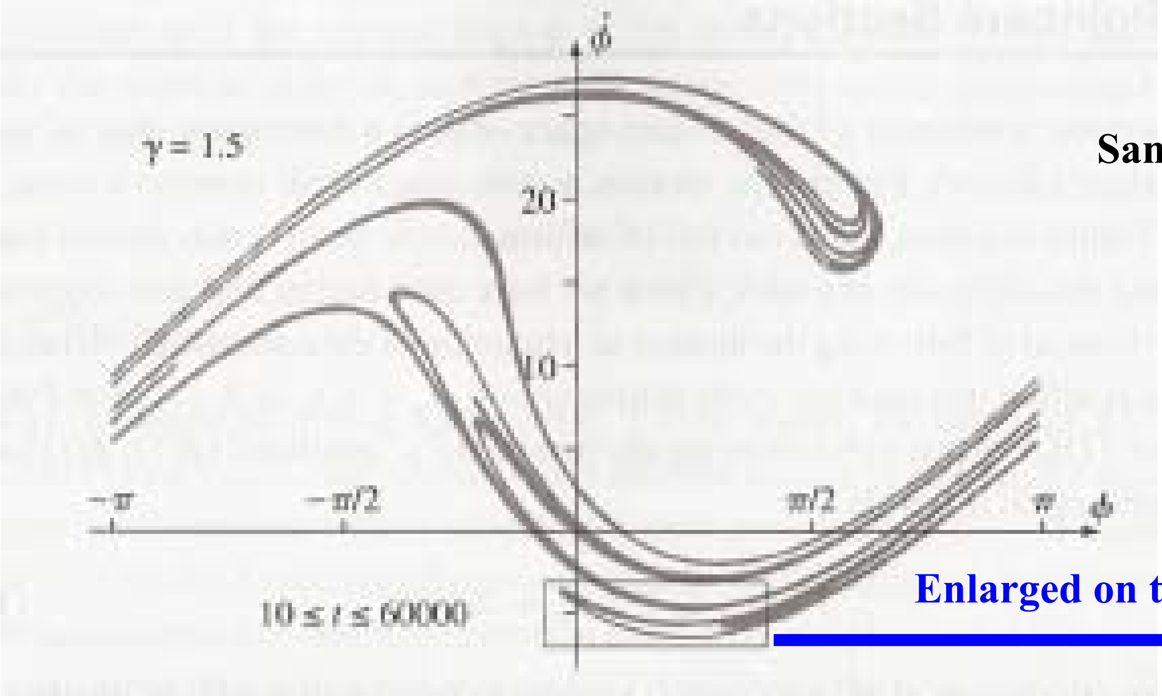
Similar to the bifurcation diagram, look at a sub-set of the data

1) **Solve for $\phi(t)$, and construct the state-space orbit**
2) **Plot a <u>periodic sampling</u> of the orbit**

$$\left[\phi(t_0),\dot{\phi}(t_0)\right],\left[\phi(t_0+1),\dot{\phi}(t_0+1)\right],\left[\phi(t_0+2),\dot{\phi}(t_0+2)\right],\ldots$$

with t_0 chosen after the attractor behavior has been achieved

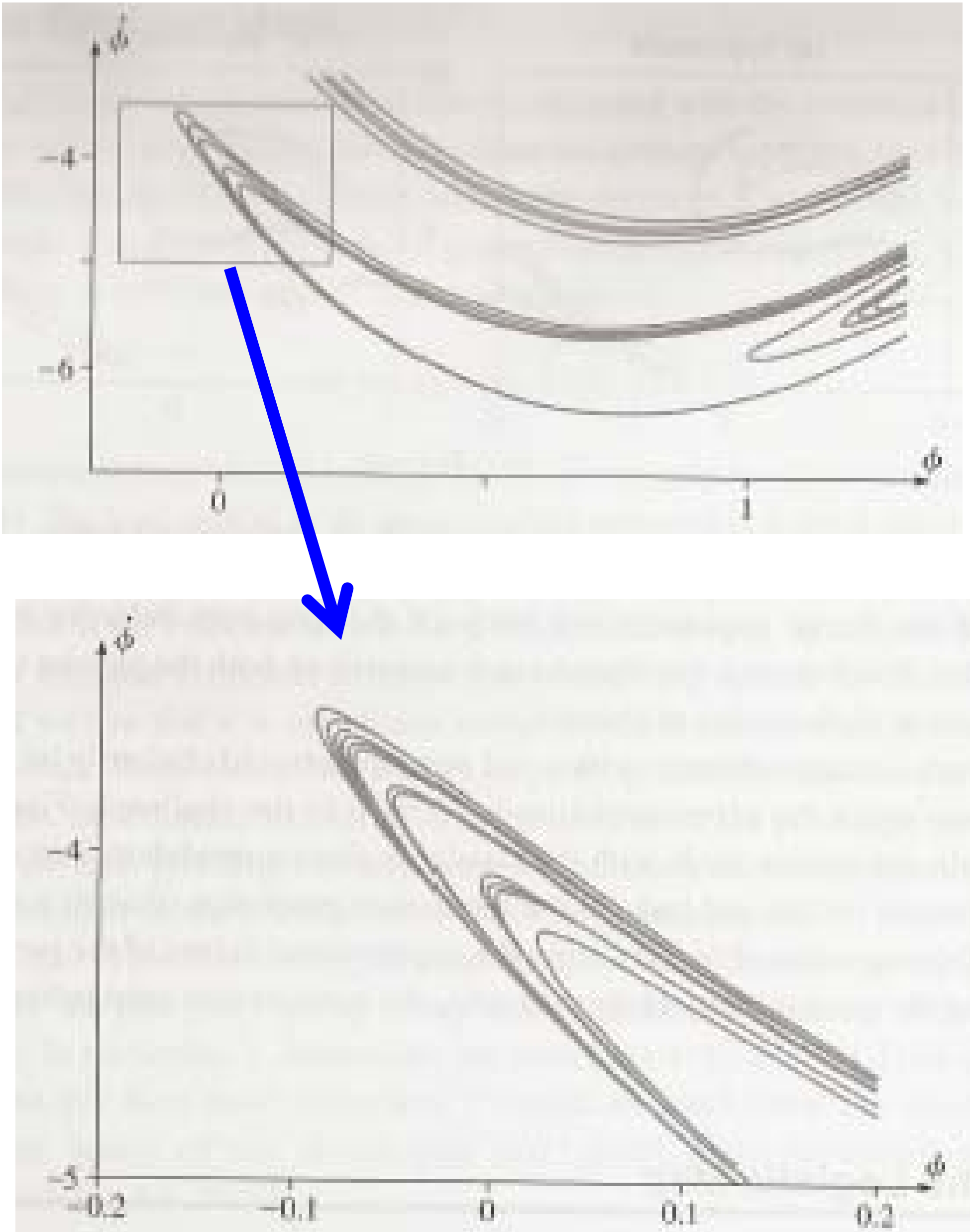
$\gamma = 1.5$
$\beta = \omega_0/8$



Samples 10 – 60,000

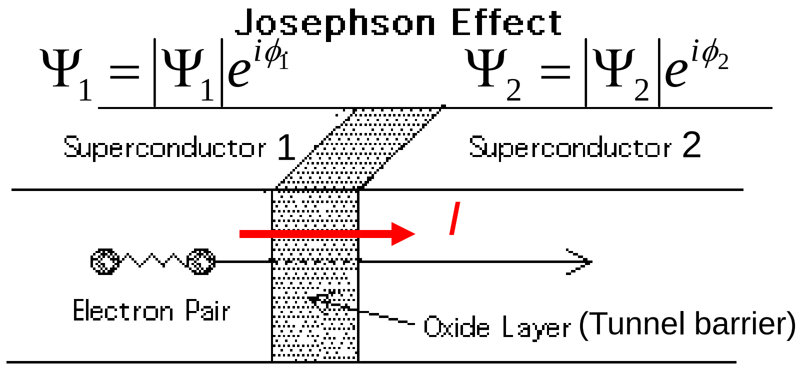
Enlarged on the next slide

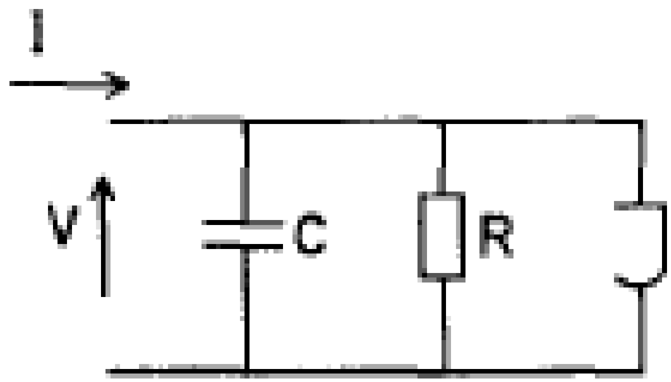
The Poincaré Section is a Fractal



The Poincaré section is a much more elegant way to represent chaotic motion

The Superconducting Josephson Junction as a Driven Damped Pendulum





$$I = I_0 \sin\phi$$

$$\frac{d\phi}{dt} = \frac{2eV}{\hbar}$$

The Josephson Equations

$\phi = \phi_2 - \phi_1$ = phase difference of SC wave-function across the junction

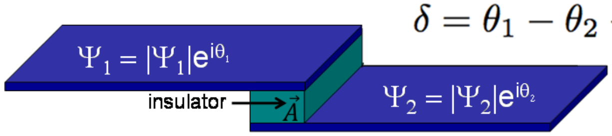
$$I_{dc} + I_{rf}\cos(\omega t) = \frac{\hbar C}{2e}\left(\ddot{\phi} + \frac{1}{RC}\dot{\phi} + \frac{2eI_0}{\hbar C}\sin\phi\right).$$

Radio Frequency (RF) Superconducting Quantum Interference Devices (SQUIDs)

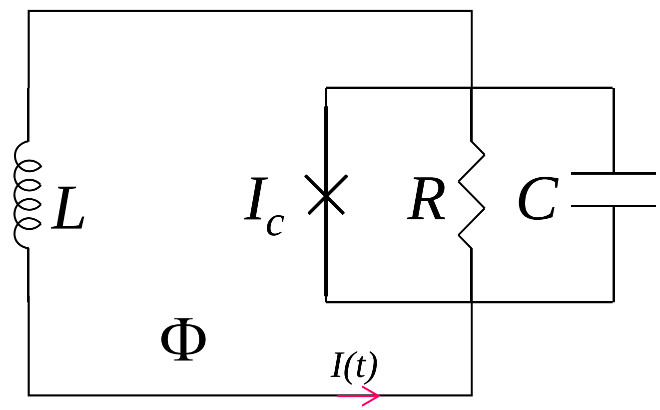
$$\Phi_{applied} + \Phi_{induced} = n\Phi_0$$

$$\Phi_{DC} + \Phi_{rf}\sin\omega t = \frac{\Phi_0 \delta}{2\pi} + L\left(I_C \sin\delta + \frac{L}{R}\frac{\Phi_0}{2\pi}\frac{d\delta}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\delta}{dt^2}\right)$$

Flux Quantization in the loop

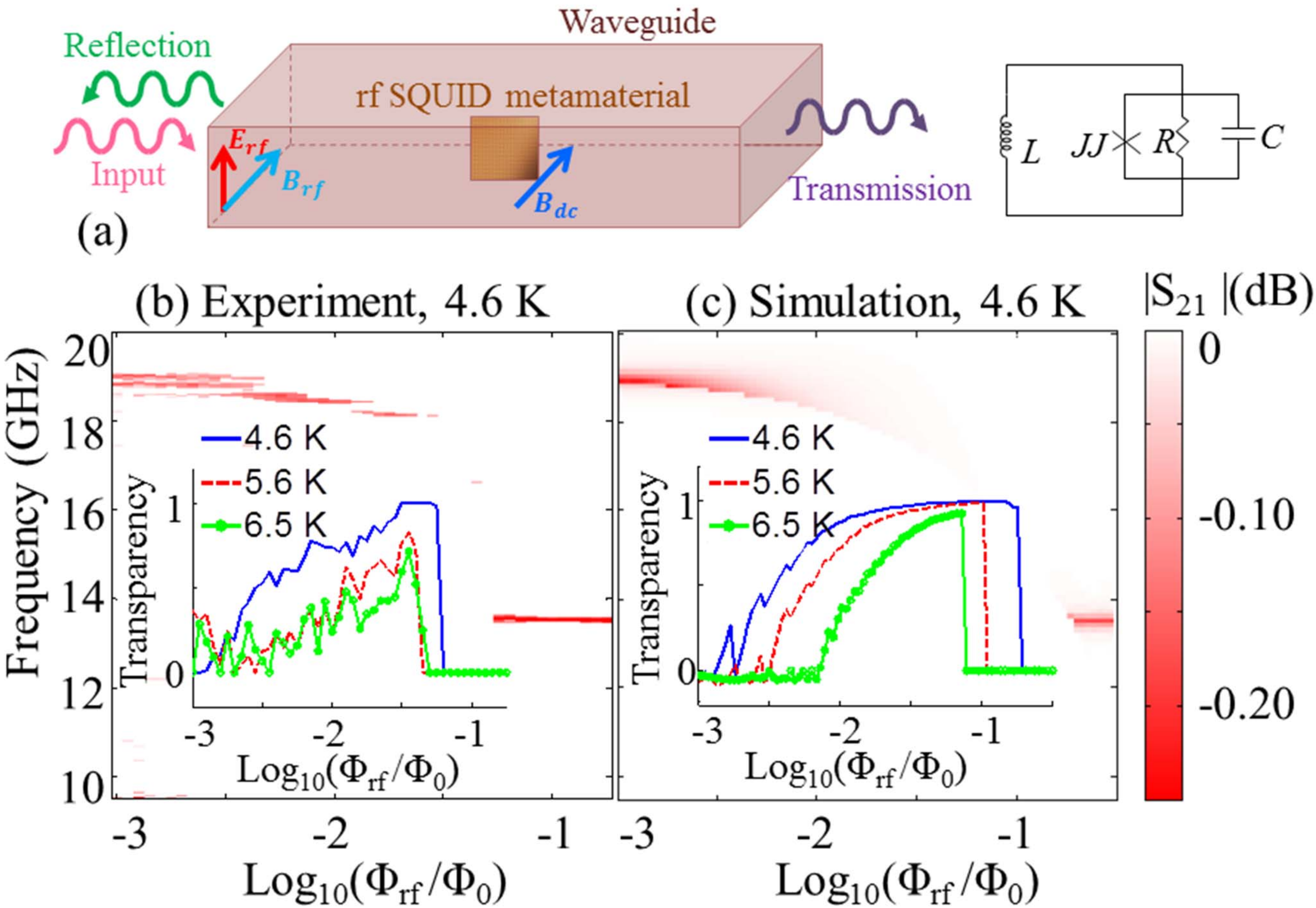


$$\delta = \theta_1 - \theta_2 - \frac{2e}{\hbar}\int_1^2 \vec{A}\cdot d\vec{l}$$

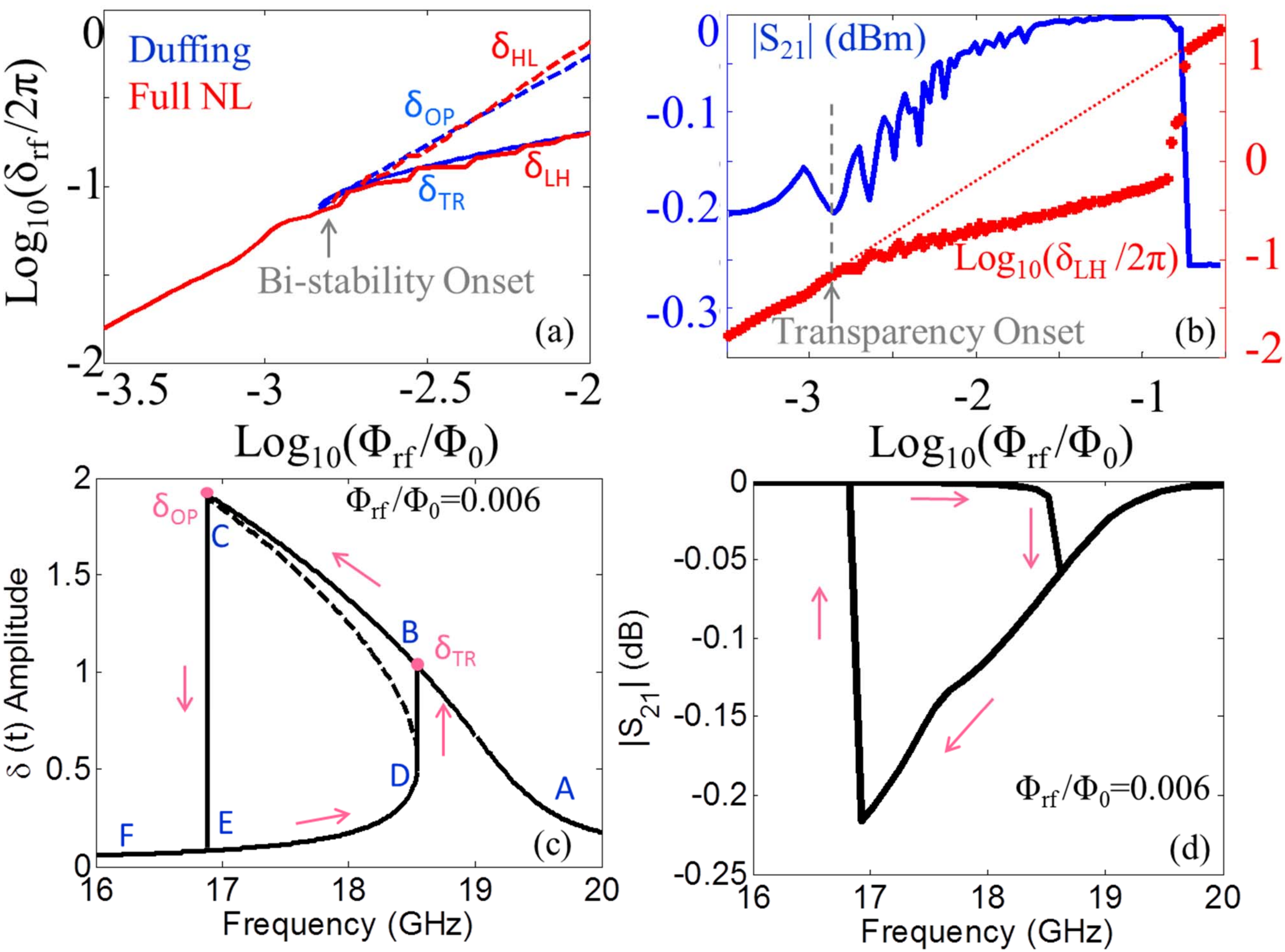


$$2\pi\left[f_{DC} + f_{rf}\sin\left(\frac{\omega}{\omega_0}\tau\right)\right] = \delta + \beta_{rf}\sin\delta + \frac{1}{RC\omega_0}\frac{d\delta}{d\tau} + \frac{d^2\delta}{d\tau^2}$$

$$\beta rf = \frac{2\pi L I_C}{\Phi_0} \qquad \omega_0 = \sqrt{\frac{1}{LC}} \qquad \tau = \omega_0 t$$



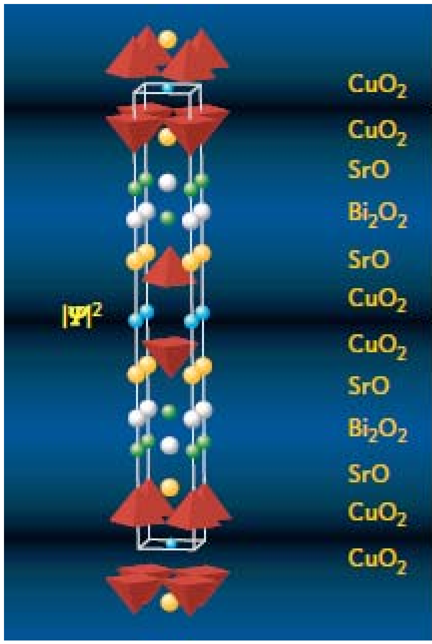
Single rf-SQUID D. Zhang, *et al*., Phys. Rev. X (in press), arXiv:1504.08301

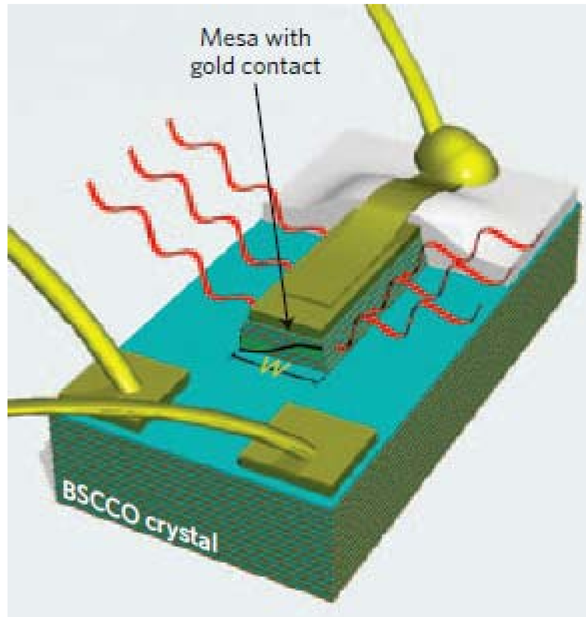


D. Zhang, *et al*., Phys. Rev. X (in press), arXiv:1504.08301

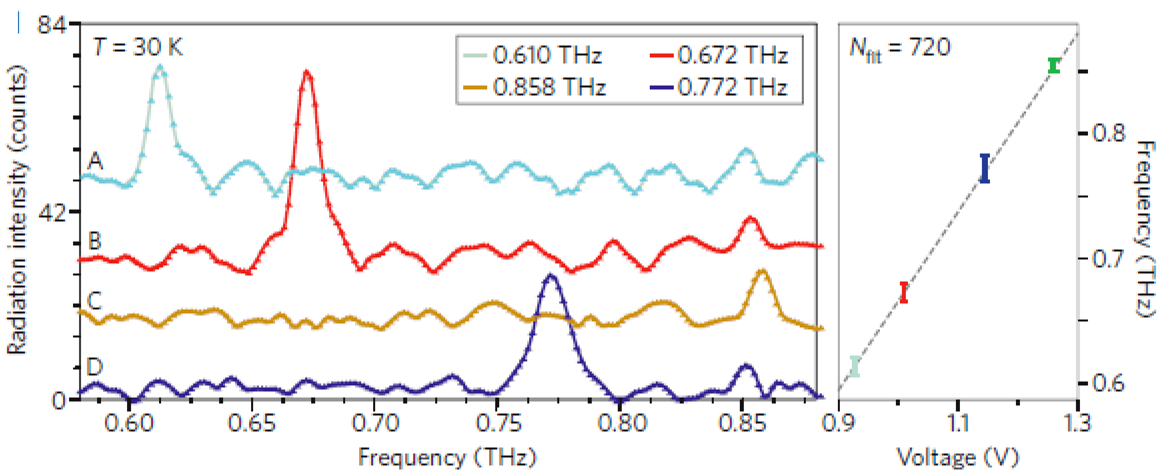
THz Emission from the Intrinsic Josephson Effect

A classic problem in nonlinear physics





L. Ozyuzer, *et al.*, Science <u>318</u>, 1291 (2007)



$$\frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V$$

$$I = I_c \sin\varphi$$

$\Phi_0 = h/2e = 2.07 \times 10^{-15}\ \mathrm{Tm}^2$

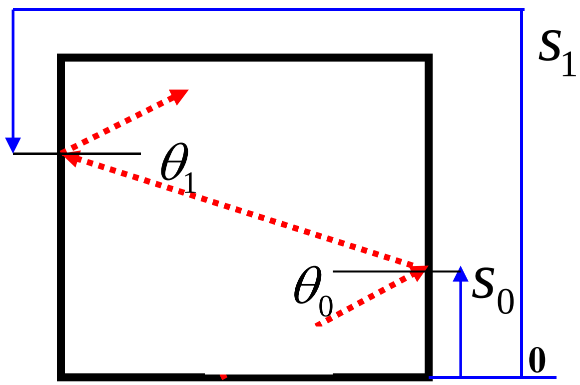
$$f_{JJ} = \frac{2e}{h} V = (0.483\ \mathrm{THz/mV}) V$$

DC voltage on junction creates an oscillating ϕ(t), which in turn creates an AC current that radiates

Best emission is seen when the crystal is partially heated above T_c!
Results are extremely sensitive to details (number of layers, edge properties, type of material, width of mesa, etc.)
Many competing states do not show emission
Emission enhanced near cavity mode resonances → requires non-uniform current injection, assisted by inhom. heating, π-phase kinks, crystal defects

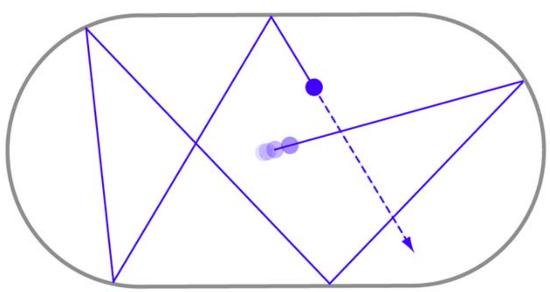
Chaos in Newtonian Billiards

Imagine a point-particle trapped in a 2D enclosure and making elastic collisions with the walls



Describe the successive wall-collisions with a "mapping function"

$$\left.\begin{aligned} s_{n+1} &= f(s_n, \theta_n) \\ \theta_{n+1} &= g(s_n, \theta_n) \end{aligned}\right\}$$ **Linear Maps for "Integrable" systems !!**



$$\left.\begin{aligned} s_{n+1} &= f(s_n, \theta_n) \\ \theta_{n+1} &= g(s_n, \theta_n) \end{aligned}\right\}$$ **Non-Linear Maps for "Chaotic" systems !!**

- **The "Chaos" arises due to the <u>shape of the boundaries</u> enclosing the system.**

Computer <u>animation</u> of extreme sensitivity to initial conditions for the stadium billiard