A NEW ANALOGY BETWEEN MECHANICAL AND ELECTRICAL SYSTEMS

By F. A. Firestone
University of Michigan

Abstract

By considering each mass in a linear mechanical system as having two terminals, one fixed in the mass and one fixed to the frame of reference, every linear mechanical system is reduced to a multiplicity of closed mechanical circuits to which force and velocity relations similar to Kirchhoff's laws, may be applied. The conventional mechanical-electrical analogy is derived from the similarity of the equations $v=f/z$ and $I=E/Z$. It is incomplete in the following respects which lead to difficulty in its application.

I. There is a lack of analogy in the use of the words "through" and "across" which indicates a fundamental difference in the nature of the analogous quantities, for instance, force through and e.m.f. across.

II. Mechanical elements in series must be represented by electrical elements in parallel, and *vice versa.*

III. Mechanical impedances in series must be combined as the reciprocal of the sum of the reciprocals while electrical impedances in series are additive.

IV. There is an incompleteness in the mechanical analogues of Kirchhoff's laws.

The new analogy is derived from the similarity of the following equations: $v=f\bar{z}$ and $E=IZ$ where $\bar{z}$ is the reciprocal of the mechanical impedance as usually defined. This new analogy is complete in all of the above-mentioned respects in which the old analogy failed. It leads to analogous relations of a simple sort and permits an equivalent electrical circuit to be drawn in an easy intuitive manner.

Introduction

If a group of physical concepts or quantities are related to each other in a certain manner, as by equations of a certain form, and another group of concepts or quantities are interrelated in a similar manner, then an analogy may be said to exist between the concepts of the one group and those of the other group. The essential element of an analogy is a similarity between the relations within one group and the relations within the other group. Two groups of concepts may be analogous in some respects and not analogous in other respects.

An analogy is valuable to the extent that it permits a knowledge of one field to be applied in another field. Since much more is known of the characteristics of electrical circuits than of certain kinds of mechanical systems, it is often valuable to discuss a mechanical system in terms of its electrical analogue. It will be shown that the conventional mechanical-electrical analogy is incomplete in certain important particulars which make it difficult to apply in practice. A new kind of mechanical-electrical analogy is set forth below which is more complete than the old and permits an equivalent electrical circuit to be drawn in a much more straightforward and common-sense manner, not requiring such careful reasoning at each step.

We have all learned at school that inductance in an electrical circuit plays a part similar to mass in a mechanical system.[1] In substantiation of this viewpoint it is mentioned that the energy stored in the magnetic field of the inductance is $\frac{1}{2}\ LI^2$ and the kinetic energy of a mass is $\frac{1}{2}\ mv^2$; that the inductance tends to prevent a change of current by generating a back e.m.f. of magnitude LdI/dT just as a mass tends to prevent a change of velocity by producing a reacting force of magnitude mdv/dt. But one might say with equal truth that capacity plays the rôle of electrical mass because the energy stored in the electrostatic field of a condenser, $\frac{1}{2}\ CE^2$, corresponds to the kinetic energy of a mass, $\frac{1}{2}\ mv^2$; and also that the condenser tends to prevent a change of e.m.f. by absorbing a current of magnitude CdE/dt just as a mass tends to prevent a change of velocity by producing a reacting force of magnitude mdv/dt.

Similarly we have been told that capacity in an electrical circuit plays a part similar to the compliance c of a spring in a mechanical system. And it is pointed out that the energy stored in the electrostatic field of the condenser is $\frac{1}{2}\ CE^2$ corresponding to the energy $\frac{1}{2}\ cf^2$ stored in a spring by a force f; that a condenser will hold a charge, CE, proportional to the e.m.f. while a spring undergoes a displacement, cf, proportional to the applied force. But it may be said with equal reason that inductance plays the part of electrical stiffness because the energy stored in the magnetic field of the inductance, $\frac{1}{2}\ LI^2$, corresponds to the energy stored by a spring, $\frac{1}{2}\ cf^2$; and also that an inductance will store a voltage impulse ($\int Edt$) of magnitude LI proportional to the current while a spring undergoes a displacement, cf, proportional to the applied force.[2]

It is, therefore, evident that a new analogy is possible in which force is identified with current rather than with e.m.f. as in the old analogy. In what follows, it will be shown that the new analogy is more complete and more useful than the old.

THE MECHANICAL ELEMENTS

We will confine our attention to the consideration of those mechanical systems wherein all the forces and the resulting velocities are essentially

[1] For instance, Starling, *Electricity and Magnetism*, p. 307.

[2] In Liven's, *Theory of Electricity*, p. 416, it is said "It must, however, be particularly emphasized that we have no definite proof that this (magnetic) energy is kinetic, it is merely a matter of convenient choice so to regard it."

in one line. Many vibrating systems, including torsional systems, can be very simply reduced to such a linear problem. The elements of a mechanical system are assumed to be of three kinds, springs, resistances and masses, although practically it may be difficult to realize any one of them in its pure form. We may assume that all of the elements lie in one horizontal line and that the positive direction along this line is from left to right. In order to understand the analogy in detail, it is necessary to look carefully at the nature of each kind of mechanical element and to be sure that we have a clear conception of the meaning of the quantities which may be used in describing its state.

A spring has two terminals to which force may be applied, and since the force is the same in all parts of the spring we will speak of the "force through the spring." The force through the spring may be assumed to be positive when the spring is under tension, and the fact that a spring is under tension may be indicated by an arrow in the positive

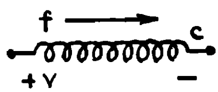


FIG. 1. *Spring.*

FIG. 2. *Resistance.*

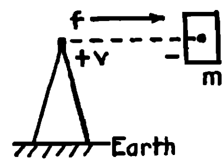


FIG. 3. *Mass.*

direction (to the right) as shown in Fig. 1. This is merely a convention which must be remembered. There will always be a positive force through a spring when it is longer than its unstrained length. If the left terminal of a spring through which there is a positive force (tension) is attached to some object, then the spring lies on the positive side of the object and the object will be subjected to a force in the positive direction. If the right terminal of the spring is attached to the object, then the spring lies on the negative side of the object, and the positive force through the spring will pull the object in the negative direction.

The relative velocity of the terminals will be called the "velocity difference across" the spring or simply the "velocity across." The velocity difference is counted positive at a given terminal if in imagining that terminal as fixed, the other terminal is moving in the positive direction. This may be indicated by + and − signs as in Fig. 1. Thus if we stand at the left end of the spring and imagine it as fixed, the plus sign by our side will indicate that the other terminal is moving in the positive direction (to the right in the figure) or away from us. Similarly, the negative at the right end indicates that if the right end is fixed, the left end will be moving in the negative direction (to the left). Consequently, a plus sign at the left terminal of a spring indicates that the spring is growing longer, and *vice versa*. (In a torsional spring there is no essential distinction between the two possible directions of twist analogous to lengthening and shortening in a linear spring. One may merely assume arbitrarily that a counterclockwise velocity of the nearer terminal relative to the farther is a positive angular velocity difference across the spring; and likewise that a counterclockwise torque on the nearer terminal sends a positive torque through the spring.)

A mechanical resistance may be visualized as two massless concentric tubes with a layer of viscous oil between. It also has two terminals. With the same conventions as in the previous paragraph and as indicated in Fig. 2, a resistance will have a positive force through it (tension) when the velocity difference across it is positive at the left terminal (resistance growing longer).

It is not obvious that a mass has two terminals. One cannot apply a force to a spring or to a resistance without grasping it at two points, but force can apparently be applied to a mass by contacting it at one point only. But a force is itself a two-terminal element since action and reaction are equal and opposite; it is impossible to push or pull without standing on something. Thus in order that a force may act on a mass, the force must react against another mass, which may be the earth. The acceleration produced on a mass by a given force is the same whether that force reacts against a small mass or against the earth. Since we measure the velocity of the mass relative to the earth,[3] and since any impressed force must in effect react against the earth, it is helpful to assume that every mass has two terminals, one of which is some arbitrary point of the mass, the other being a point near the mass, which is fixed to the earth, as shown in Fig. 3. Here again the velocity difference across the mass (relative velocity) of its terminals is considered positive at the left terminal when the right terminal is moving to the right. (While in ordinary language we speak of the velocity *of* the mass instead of the velocity difference *across* the mass, this is similar to the case of any two terminal electrical element, one side of which is grounded, wherein the potential *of* the high side is the same as the potential difference *across* the element.) If we impress on a mass an external positive force f' as shown in Fig. 4, the mass will receive an acceleration which will be positive at the left terminal and which will call forth a reactive tensile force between the terminals of the mass

[3] Any unaccelerated frame of reference would do equally well.

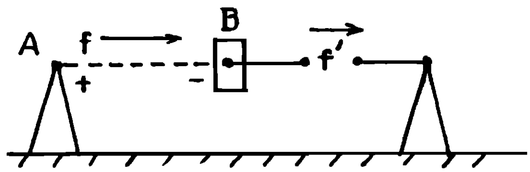
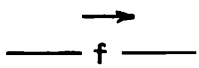
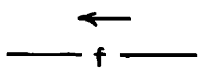


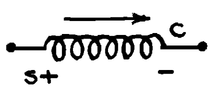
FIG. 4.



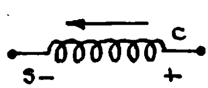
A tensile force tending to pull these terminals together.



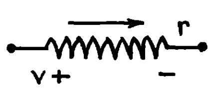
A compressive force tending to push these terminals apart.



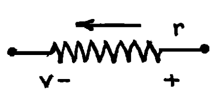
A spring in tension. The displacement differences must therefore be positive at the left terminal, meaning that as seen from the left terminal the right terminal must have moved in the positive direction. Spring lengthened.



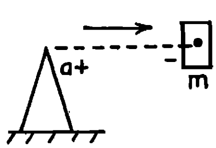
A spring in compression. The displacement difference is therefore negative at the left terminal. Spring shortened.



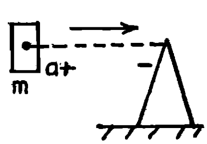
Resistance in tension. The velocity difference must therefore have the sign shown, resistance growing longer.



Resistance in compression, growing shorter.



Mass with acceleration difference of signs shown must be in tension. The right terminal is accelerating toward the right.



This mass with acceleration difference as shown must be in tension. The left terminal is accelerating to the left.

FIG. 5.

A and B, of magnitude $f = ma$. Thus the tensile force f' has produced a tensile force through the mass to the earth.[4] It is as if the earth[4] did not like to see masses accelerated and acted on them with a restraining force. Assuming as before that a tensile force through an element is positive, that is, the element is trying to pull its terminals together, there is a positive force through a mass when there is an acceleration across it which is positive at the left terminal. Similar conventions can be worked out for a moment of inertia in a torsional system.

The above conventions are summarized and exemplified in Fig. 5.

Mechanical elements will be assumed to be connected mechanically in series when they are joined end to end as shown in Fig. 6. In such an arrangement is it obvious that the force through all the elements is equal; if one element is in tension they are all in tension. Likewise the velocity difference across a series of elements is the algebraic sum of the velocity

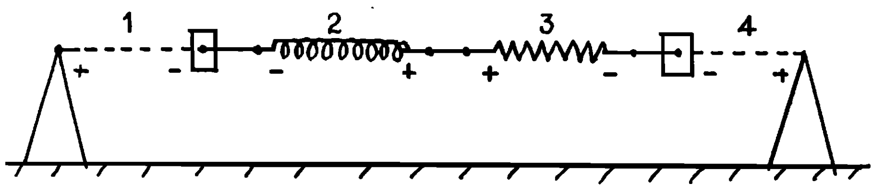


FIG. 6.

differences across each element. When a number of mechanical elements are connected to two common junction points, as shown in Fig. 7, they may be said to be connected in parallel. The connecting bar on the right is assumed to remain vertical. It is obvious that in such an arrangement the velocity difference across all of the elements is the same. Furthermore the total force through the combination is the algebraic sum of the separate forces through the elements.

Springs and resistances may be connected either in series or in parallel but a number of masses can be connected in parallel only, as one terminal of each mass is connected to the earth. Two masses only may be connected in series through other elements as shown in Fig. 6, but if they were connected directly together, they would in effect be in parallel. Of course, springs and resistances may be connected either in series or in parallel with masses.

We may state two additional relations which hold for any linear mechanical system, the force law and the velocity law. According to the force law, the algebraic sum of all the forces acting on any junction

[4] Or other frame of reference.

of mechanical elements is zero. For instance if we consider the central junction in Fig. 7 and assume that a force whose arrow is away from the junction should be counted positive, then

$$f - f_1 - f_2 - f_3 - f_4 = 0.$$

The velocity law states that the algebraic sum of the velocity differences around any closed mechanical circuit is zero. It is necessary to take as the sign of each velocity difference the first sign which is seen on approaching the element in going around the circuit. Thus in Fig. 6 if the signs of the velocities are assumed as shown,

$$v_1 - v_2 + v_3 - v_4 = 0.$$

Also in Fig. 10 the velocity law applied around the central circuit would give $v_3+v_4+v_5-v_6=0$.

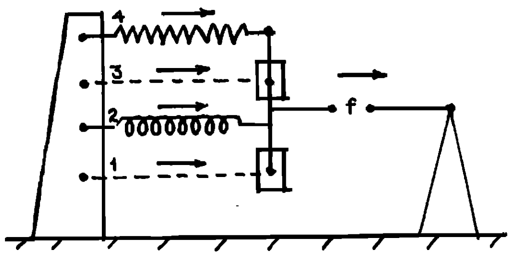


FIG. 7.

We follow the usual convention in defining mechanical impedance as the complex quotient of the force through and the velocity difference across an element or combination of elements.

$$z = f/v.$$

(This definition was originally chosen with the conventional analogy in mind.) In a series of mechanical elements having individual impedances of z_1, z_2, z_3, etc., the impedance of the combination would be

$$z = \frac{f}{v_1 + v_2 + v_3 + +} = \frac{1}{\frac{v_1}{f} + \frac{v_2}{f} + \frac{v_3}{f} + +} = \frac{1}{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + +}.$$

Thus mechanical impedances in series add as the reciprocal of the sum of the reciprocals as do electrical impedances in parallel. Similarly if we have a number of mechanical impedances in parallel, the impedance of the combination is

$$z = (f_1 + f_2 + f_3 + +)/v = z_1 + z_2 + z_3 + + .$$

So mechanical impedances in parallel are additive like electrical impedances in series.

In the above description of the mechanical elements and the mechanical circuit no detailed assumption has been made as to the nature of any analogy with the electrical circuit which may be noted later, although some of the conventions were suggested by analogy. By considering each mass as having one terminal fixed to the frame of reference, every mechanical system is reduced to a multiplicity of closed mechanical circuits, thereby preparing the way for the application of electrical analogies.

THE CONVENTIONAL ANALOGY

The conventional mechanical-electrical analogy may be derived from the fact that for most mechanical systems, an electrical system can be invented of such a sort that the differential equations of motion in the two systems, as expressed in terms of displacement and charge, respectively, will be of the same form. If impedances are defined as in the following vectorial equations

$$= f/z \text{ (mechanical)} \quad I = E/Z \text{ (electrical)}$$

the form of the impedances as derived from the differential equations will be similar and will justify the following conventional analogy.

THE CONVENTIONAL ANALOGY

Mechanical	*Electrical*
Force through $= f$ (dynes)	e.m.f. across $= E$ (volts)
Velocity across $= v$ (cm/sec.)	Current through $= I$ (amperes)
Displacement across $= s$ (cm)	Charge through $= Q$ (coulombs)
Impulse through $= p$ (dyne sec.)	Voltage impulse $= \int E dt$ (volt sec.)
Impedance $= z$ (ohms)*	Impedance $= Z$ (ohms)
Resistance $= r$ (ohms)	Resistance $= R$ (ohms)
Reactance $= x$ (ohms)	Reactance $= X$ (ohms)
Mass $= m$ (grams)	Inductance $= L$ (henries)
Compliance $= c$ (cm/dyne)	Capacity $= C$ (farads)
Power $= f\,v$ (ergs/sec.)	Power $= EI$ (watts)
z of resistor $= r$ (ohms)	Z of resistor $= R$ (ohms)
z of mass $= i\omega m$ (ohms)	Z of inductance $= i\omega L$ (ohms)
z of spring $= -i/\omega c$ (ohms)	Z of condenser $= -i/\omega C$ (ohms)
Impedances in series $z = \dfrac{1}{1/z_1 + 1/z_2 + 1/z_3 +}$	Impedance in series $Z = Z_1 + Z_2 + Z_3 +$
Impedances in parallel $z = z_1 + z_2 + z_3 +$	Impedances in parallel $Z = \dfrac{1}{1/Z_1 + 1/Z_2 + 1/Z_3 +}$
Force and velocity laws	Kirchhoff's laws
Sum of velocity differences around closed circuit is zero	Sum of currents to a junction is zero
Sum of forces to a junction is zero	Sum of e.m.f.'s around a mesh is zero

* Mechanical ohm = dyne/kine.

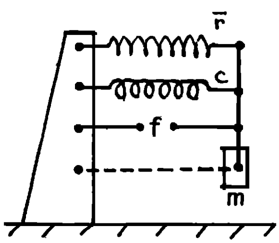
Thus in the familiar problem of the forced vibration of an elastically bound mass with friction, and its analogous circuit, shown in Fig. 8, the differential equations are

$$\frac{md^2s}{dt^2}+\frac{rds}{dt}+\frac{s}{c}=fe^{i\omega t}\quad \frac{Ld^2Q}{dt^2}+\frac{RdQ}{dt}+\frac{Q}{C}=Ee^{i\omega t}.$$

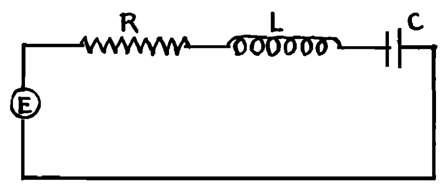
The steady state solutions of these in vector forms are

$$v=\frac{f}{r+i\left(\omega m-\dfrac{1}{\omega c}\right)}=\frac{f}{z}\quad I=\frac{E}{R+i\left(\omega L-\dfrac{1}{\omega C}\right)}=\frac{E}{Z}\cdot$$

Therefore the velocity across the mass due to the impressed force is the same as the current through the inductance due to the e.m.f. E.



Mechanical system



Conventional analogue

FIG. 8.

However, while the mechanical elements are connected in parallel, the analogous electrical elements must be connected in series. This must always be the case with this analogy since mechanical elements in parallel have a common velocity difference across them and electrical elements in series have a common current through them. In general, mechanical elements in series are represented by electrical elements in parallel, and *vice versa*, so that in a complicated mechanical system the analogous electrical elements must be placed in a manner quite contrary to intuition. A mass, which always has one terminal on the earth, will usually be represented by an inductance in the high side of the line, and a spring which is in series with the mechanical circuit and must transmit all the force from one part of the mechanical system to another will be shown as a condenser across the line with one side grounded, while a spring which connects part of the mechanical system to the earth will be represented by a condenser in series with the high wire. This is shown in Fig. 12.

The difficulty is also indicated in the use of the words "through" and "across" in the table of analogous quantities above. Force has a "through" character like current; the force through each of a series of mechanical elements is the same, just as the current through each of a series of electrical elements is the same. Also, velocity difference has an "across" character like potential difference; the velocity difference across each of a number of mechanical elements in parallel is equal, just as the potential differences across electrical elements in parallel are equal. It is in ignoring these fundamental characteristics of the analogous quantities and placing "force through" analogous to "e.m.f. across" that the conventional analogy becomes unnecessarily difficult of application.

Furthermore it is noted in the table that the laws for the addition of impedances are not analogous, and also that there is a lack of accurate correspondence between Kirchhoff's laws and their mechanical analogues.

THE NEW ANALOGY

The analogy here proposed may be derived from the similarity of the equations

$$v = f\bar{z} \quad E = IZ$$

where $\bar{z}$ is the bar impedance, the reciprocal of the mechanical impedance. The real part of $\bar{z}$ is the bar resistance $\bar{r}$, and its imaginary part, the bar reactance $\bar{x}$. Since the bar impedance of a spring of compliance c is $i\omega c$ and the impedance of an inductance L is $i\omega L$, the impedance of an inductance $L=c$ is at all frequencies equal to the bar impedance of the spring. Similarly the far impedance of a mass m is $-i/\omega m$ and the impedance of a condenser is $-i/\omega c$ so that the impedance of a capacity $C=m$ will at all frequencies be equal to the bar impedance of the mass. The bar impedance of a mechanical resistance r ohms is $1/r$ and is therefore equal to the impedance of an electrical resistance $R=1/r$ ohms. In this way the following analogy is established.

THE NEW ANALOGY

Mechanical	*Electrical*
Force through $=f$ (dynes)	Current through $=I$ (amperes)
Velocity across $=v$ (cm/sec.)	e.m.f. across $=E$ (volts)
Displacement across $=s$ (cm)	Voltage impulse across $=\int Edt$ (volt sec.)
Impulse through $=p$ (dyne sec.)	Charge through $=Q$ (coulombs)
[a]Bar impedance $=\bar{z}$ ($\overline{\text{ohms}}$)*	Impedance $=Z$ (ohms)

[a] Bar impedance = velocity across/force through.

* $\overline{\text{ohms}}$ = bar ohms = kines/dyne.

THE NEW ANALOGY (*Cont'd*)

Mechanical	*Electrical*
[b]Bar resistance $=\bar{r}$ ($\overline{\text{ohms}}$)	Resistance $=R$ (ohms)
[c]Bar reactance $=\bar{x}$ ($\overline{\text{ohms}}$)	Reactance $=X$ (ohms)
Mass $=m$ (grams)	Capacity $=C$ (farads)
Compliance $=c$ (cm/dyne)	Inductance $=L$ (henries)
Power $=fv$ (ergs/sec.)	Power $=IE$ (watts)
$\bar{z}$ of resistor $=\bar{r}$ ($\overline{\text{ohms}}$)	Z of resistor $=R$ (ohms)
$\bar{z}$ of mass $=-i/\omega m$ ($\overline{\text{ohms}}$)	Z of condenser $=-i/\omega C$ (ohms)
$\bar{z}$ of spring $=i\omega c$ ($\overline{\text{ohms}}$)	Z of inductance $=i\omega L$ (ohms)
Bar impedances in series $\bar{z}=\bar{z}_1+\bar{z}_2+\bar{z}_3+$	Impedances in series $Z=Z_1+Z_2+Z_3+$
Bar impedances in parallel $\bar{z}=\dfrac{1}{1/\bar{z}_1+1/\bar{z}_2+1/\bar{z}_3+}$	Impedances in parallel $Z=\dfrac{1}{1/Z_1+1/Z_2+1/Z_3+}$
Force and velocity laws	Kirchhoff's laws
Sum of forces to a junction is zero	Sum of currents to a junction is zero
Sum of velocity differences around closed circuit is zero	Sum of e.m.f.'s around a mesh is zero

[b] Bar resistance = real part of bar impedance.

[c] Bar reactance = imaginary part of bar impedance.

It has seemed advisable to introduce a new term, bar impedance, which is equal to *velocity across/force through*. It is natural that the old analogists, having arrived on the ground first, should have chosen to define impedance as *force/velocity* since that fitted in with the other assumptions they had made. But in the author's opinion, all of their assumptions were unwise and led to the left-handed result that while electrical impedances in series are additive, mechanical impedances in series must be added as the reciprocal of the sum of the reciprocals as was shown above. It would have been better if this new analogy had been thought of first, for in that case the quantity which we have been forced to call "bar impedance" would have been called "impedance" and would have been subject to the same laws of addition as are found in the electrical circuit. It is now too late to change suddenly the unfortunate definition of impedance which has been so much used in the past, so it is recommended that the term "bar impedance" be used. Then if the new analogy should prove popular, the time may come when the old definition of impedance will have fallen into disuse, at which time the "bar impedance" may be shortened to "impedance" with the new definition.[5]

[5] Of course, bar impedance is the same as mechanical admittance as usually defined, but to use the latter term would spoil the analogy and cause confusion.

This nomenclature therefore prepares the way for an evolutionary change of definition. In the meantime $\bar{z}$ will indicate that this quantity is z below the line, that is, the reciprocal of z as ordinarily defined. It should be noted, however, that $\bar{r}$ is the real part of $\bar{z}$ and is equal to r/z^2, being the reciprocal of r only when $z=r$, a pure resistance; similarly for $\bar{x}$.

The unit in which bar impedance is measured is the bar ohm ($\overline{\text{ohm}}$) which is one kine per dyne. In time the bar might be dropped thereby constituting a change of definition.

In a series mechanical system, the bar impedance of the combination is the sum of the bar impedances of the elements.

$$\bar{z} = \frac{v_1 + v_2 + v_3 + \cdots}{f} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \cdots .$$

Similarly for a parallel mechanical system

$$\bar{z} = \frac{v}{f_1 + f_2 + f_3 + \cdots} = \frac{1}{1/\bar{z}_1 + 1/\bar{z}_2 + 1/\bar{z}_3 + \cdots} .$$

Thus the bar impedance follows the same laws of addition as electrical impedance.

The new electrical analogue for displacement is the voltage impulse,[6] ($\int E dt$) this being the electrical quantity which is measured by a ballistic galvanometer. It was given this name by the old analogists because to them it was analogous to mechanical impulse. The lack of correspondence at this point in the new analogy is again a mere matter of definition. In any problem where a simple harmonic displacement amplitude is specified, it is convenient in the new analogy to convert it to velocity by multiplying by $i\omega$.

In both analogies, mechanical power and electrical power are analogous.

It will be noted that in this analogy the words "through" and "across" are analogously used. In the old analogy it was appealing to have force and electromotive force be analogues because of the similar sound of the words which, when loosely used, appear to have a similarity of meaning; but when one analyzes the case more closely and finds that force *through* is analogous to electromotive force *across*, some of the attractiveness is lost. One may also favor the old analogy if he thinks of e.m.f. as cause and current as effect, of force as cause and velocity as

[6] Bennet and Crothers, *Introductory Electrodynamics for Engineers*, p. 470.

effect. However, in a series electrical circuit it is more convenient to consider the current through an element as the cause and the e.m.f. across the element as the effect, and in a series mechanical system, of force through an element as cause and velocity across the element as effect, thereby in that case favoring the new analogy.

With this analogy also, the differential equations of motion in the analogous systems will be of the same form if written in appropriate variables. For instance, consider the problem in Fig. 9 where a velocity

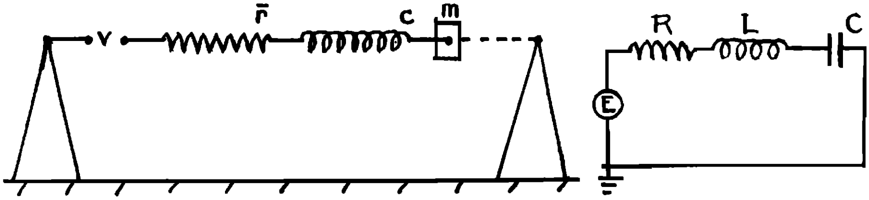


FIG. 9.

$ve^{i\omega t}$ is impressed in series with a resistance, spring, and mass and we wish to find the resulting force through the system. Applying the velocity law around the mechanical circuit and using the impulse $p=\int f dt$ as dependent variable we get the following differential equations:

$$c\frac{d^2p}{dt^2}+\bar{r}\frac{dp}{dt}+\frac{p}{m}=ve^{i\omega t};\quad \frac{Ld^2Q}{dt^2}+\frac{RdQ}{dt}+\frac{Q}{C}=Ee^{i\omega t}.$$

Since $f=dp/dt$ and $E=dQ/dt$ the steady state solutions of these equations in vectorial form are

$$f=\frac{v}{\bar{r}+i(\omega c-1/\omega m)}=\frac{v}{\bar{z}};\quad I=\frac{E}{R+i(\omega L-1/\omega c)}=\frac{E}{Z}.$$

The velocity across any element can be found by multiplying the above value of f by the bar impedance of the element.

In the above differential equations and their solutions the analogy as set forth in the above table is obvious. In the series mechanical circuit shown, the impulse through each element is the same while the displacements across them are different, consequently it is more convenient to write the differential equation with impulse as dependent variable, than displacement. We may therefore say that if the founders of vibration theory had happened to consider the forced vibration problem of the series mechanical system with impressed velocity, instead of the parallel system with impressed force, the analogy here presented is the one which they would have noted as being the most obvious and appealing.

EXAMPLES

The above differential equations have been presented for the benefit of those who feel that an analogy should come from a similarity of differential equations. However, when one wishes to find the steady state e.m.f.'s and currents in a given electrical circuit he does not write down any differential equations. He finds the impedances of the elements at the given frequency by well-known formulae, applies Kirchhoff's first law to the currents entering the junction points, applies Kirchhoff's second law to the voltages around each mesh, and solves the simultaneous equations thus derived for the unknown currents and voltages. It is, therefore, desirable that one should not have to write the differential equations for either the mechanical or the electrical system in finding the electrical analogue of a given mechanical system.

In the new analogy, the equivalent electrical circuit is drawn to resemble the original mechanical system, remembering that one terminal of each mass is connected to earth. Each spring is represented by an inductance $L=c$; each mass is represented by a capacity $C=m$ one side of which is connected to ground; each mechanical resistance of bar resistance $\bar{r}$ is replaced by an electrical resistance $R=\bar{r}$. Mechanical elements in series are represented by electrical elements in series; parallel mechanical elements by parallel electrical elements. If there is any question as to the validity of the equivalent circuit, Kirchhoff's laws may be applied to the junctions and meshes of the circuit and these should yield equations which are equivalent to those which will be obtained on applying the force law and the velocity law to the mechanical system. Kirchhoff's first law for the currents flowing to any junction is quite analogous to the force law for the forces acting on any junction; Kirchhoff's second law for the e.m.f.'s around a mesh is analogous to the velocity law for the velocities around a closed mechanical circuit.

Having arrived at this much of an understanding of the mechanical problem we might consider abandoning the analogy and working the mechanical problem directly by an application of the force and velocity laws. In case it is necessary to write down Kirchhoff's equations for the electrical circuit and to solve them, little is gained by drawing the equivalent circuit; the analogous equations can be written down at once for the mechanical system. Many well-known electrical laws as, for instance, the reciprocal theorem and Thevenin's theorem, can be converted by the analogy into equivalent mechanical laws and used directly. In this manner, the electrical technique for the solution of problems can be used without converting the problem to an electrical circuit. In case the properties of the analogous electrical circuit have already been worked out so that no equations need be written or solved, then there is a decided advantage in working with the electrical analogue.

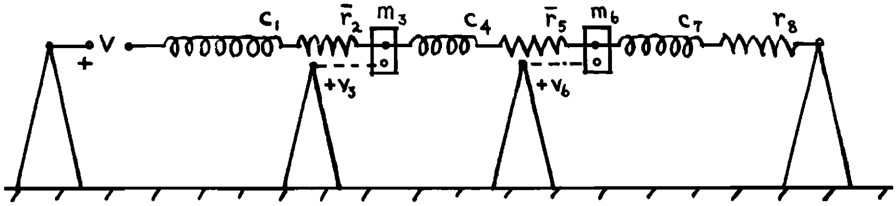


FIG. 10. *Mechanical system.*

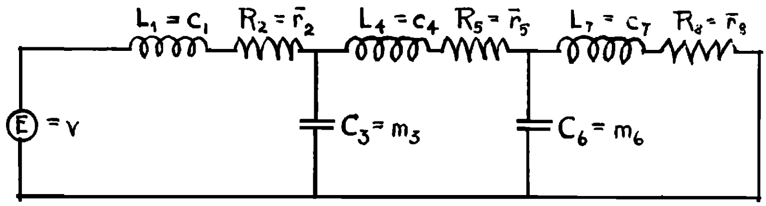


FIG. 11. *New analogue.*

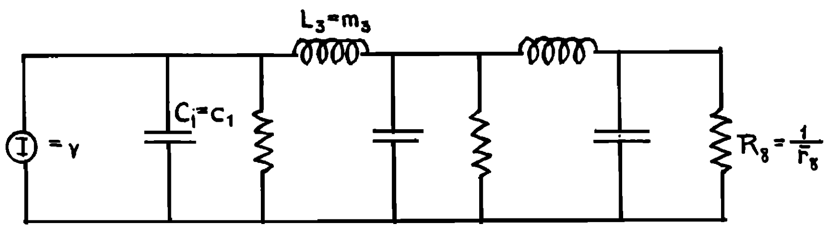
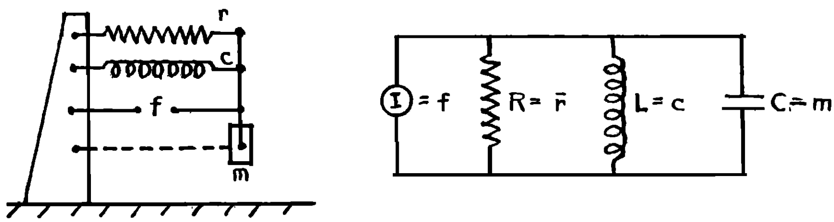


FIG. 12. *Conventional analogue.*

As an example consider the linear mechanical system shown in Fig. 10 in which an impressed velocity is operating and we might wish to know the velocity across (or of) m_6. The new analogue, derived as explained above, is shown in Fig. 11, and the electrical circuit looks much like the mechanical system. The e.m.f. across c_6 will be the desired velocity across m_6. The conventional analogue is shown in Fig. 12 and the circuit is just the opposite of what one would at first glance expect. C_1 and R_2 must be connected in parallel because c_1 and $\bar{r}_2$ have the same force through them. L_3 must be in series with the high wire even though m_3 has one terminal to earth because the force through m_3 is the difference between the forces through c_4 and c_1. The current through L_6 is the desired velocity across m_6.

Or suppose we wish to solve the well-known problem of the forced vibration of a mass, spring, and resistance in parallel, as shown in Fig. 13, wishing to find the velocity across the mass. The new analogue is a



Mechanical system *New analogue*

FIG. 13.

constant current generator paralleled by a resistance, inductance and capacity, and our analogous problem is to find the e.m.f. across the condenser C.

$$E = IZ = I\frac{1}{1/R + 1/i\omega L - \omega C/i} = \frac{I}{1/R + i(\omega C - 1/\omega L)}\,.$$

Now putting in the mechanical values of the analogous electrical quantities we have

$$v = \frac{f}{1/\bar{r} + i(\omega m - 1/\omega c)} = f\bar{z}$$

which is recognized as being the correct result when we remember that $\bar{r}$ is the reciprocal of the mechanical resistance as usually defined.

THE NEW ANALOGUE OF ELECTROMECHANICAL DEVICES

In treating electromechanical devices, such as microphones and loudspeakers, by converting their mechanical parts to analogous electrical structures which are to be suitably coupled to the electrical system, it must be remembered that in the analogies set forth above, the c.g.s. units were used on the mechanical side while practical units were used on the electrical side. This results in power in ergs per second being analogous to power in watts, thus differing by a factor of 10^7. As long as we deal in analogies only, there is no harm in having the powers in the analogous electrical circuit be 10^7 times the powers in the mechanical system; but when an electrical circuit is coupled to an analogue circuit, the energy leaving the electrical circuit must equal the energy entering the analogue circuit. One way to resolve this difficulty is to use as our unit of force $(10^7)^{1/2}$ dynes, analogous to one ampere; and as our unit of velocity $(10^7)^{1/2}$ kines, analogous to one volt. The use of these large units will not change the magnitudes of our mechanical impedances, which involve the ratio of force and velocity, but will cause the power, which is the product of force and velocity, to be expressed in watts.

Consider for example a generator connected to a telephone receiver whose diaphragm may be idealized as a mass connected through a spring to the receiver case which, for simplicity, is considered as stationary. There is a two-way interaction between the electrical circuit and this mechanical system: a current through the magnet produces a force on the diaphragm, whose amount in dynes per ampere is assumed known; also any motion of the diaphragm generates an e.m.f. in the magnet circuit, whose amount in volts per kine need not be known. This coupling between the electrical circuit and the analogue circuit can be represented by an ideal transformer whose primary is in series with the electrical circuit and whose secondary is in parallel with the condenser and inductance which represent the effective mass and compliance of the diaphragm. The ratio of primary to secondary turns is the force factor of the magnet system as expressed in large dynes ($(10^7)^{1/2}$ dynes) per ampere. The resulting circuit may be solved in the usual manner to find the voltage across the condenser; this will be the velocity of the diaphragm expressed in large kines ($(10^7)^{1/2}$ kines).

If the electromechanical coupling is electrostatic instead of electromagnetic, then voltage in the electrical circuit results in a force on the mechanical system, that is, it results in a current in the analogue circuit. Also the velocity of the mechanical system generates a current in the electrical circuit. This calls for an inverse transformer to couple the electrical circuit and the analogue circuit of such a nature that the current through the secondary is a constant times the voltage across the primary while the current through the primary is a constant times the voltage across the secondary. This is not a familiar device but it can be replaced by an ideal transformer if either the circuit connected to its primary or to its secondary be replaced by an inverse network with respect to one ohm.[7] Changing a new analogue to its inverse network with respect to one ohm is equivalent to a change to the old analogue. In this problem of electrostatic coupling, the old analogy is advantageous

[7] See Shea, *Transmission Networks and Wave Filters*, p. 311, or Johnson, *Transmission Circuits for Telephonic Communication*, p. 230.

to the extent that it permits an ideal transformer to be used in coupling the electrical circuit to the analogue circuit.

THE ACOUSTICAL-ELECTRICAL ANALOGY

Acoustical transmission systems are not analogous to mechanical systems in quite such a simple manner as one might at first thought suppose. In such an acoustical transmission system as an acoustical filter, each main conducting tube has as its electrical analogue, an electrical line (two wire) having two input and two output terminals.[8] If a tube is used as a side branch with either a closed or an open end, it is as if the input terminals of the equivalent line were connected to the main conducting line while the output terminals were open circuited or short circuited.

The two variables usually used in the discussion of an acoustical transmission system are the sound pressure p at a surface and the volume velocity V through the surface. The acoustic impedance, Za, on a given surface, is defined as the complex quotient of p and V. Thus from the similarity of the following equations the conventional acoustical-electrical analogy may be derived.

$$V = p/Z_a \quad I = E/Z.$$

However, by defining the acoustical bar impedance as V divided by p we could also write

$$V = p\bar{z}_a \quad E = IZ$$

thereby obtaining a new analogy.

Between the two mechanical-electrical analogies previously mentioned, it was possible to make a choice as to the one which was most complete. This was possible because it is easy to tell when mechanical elements are in series and when in parallel; also a strict analogue to the Kirchhoff relations was found with the one analogy only. But if a number of acoustical transmission tubes terminate or originate in a common junction point, there is no *a priori* criterion as to whether they are connected in series or in parallel within the junction. The case is similar to a number of electric lines entering a common junction box; they may be connected either in series or in parallel within the box. If one first assumes the conventional analogy, then he will say that the tubes are in parallel within the junction because they are subject to a common sound pressure; but if he assumes the new analogy, he will

[8] W. P. Mason, B.S.T.J. 6, 258 (1927).

say that they are in series, for the same reason. There is some justification for a preference of the conventional acoustical-electrical analogy in that the sum of the volume displacements to any junction is zero analogous to Kirchhoff's second law for the e.m.f.'s around a mesh. On the other hand, the new analogy seems more rational in that with it, a tube having an open end is represented by an open circuited line, while a tube with a closed end is represented by a short circuited line; in the old analogy these relations are reversed.

Thus for acoustical systems, the two analogies seem about equally good.

CONCLUSION

The conventional mechanical-electrical analogy is incomplete in the following respects:

I. There is a lack of analogy in the use of the words "through" and "across" which indicates a fundamental difference in the nature of the analogous quantities, for instance, force through and e.m.f. across.

II. Mechanical elements in series must be represented by electrical elements in parallel, and *vice versa*.

III. Mechanical impedances in series must be combined as the reciprocal of the sum of the reciprocals while electrical impedances in series are additive.

IV. There is an incompleteness in the mechanical analogues of Kirchhoff's laws.

The new analogy is free from the above difficulties, which makes it much easier to apply and understand. By considering each mass as having one terminal fixed to the frame of reference, every mechanical system is reduced to a multiplicity of closed mechanical circuits thereby preparing the way for the application of the electrical analogy. In the new analogy, the equivalent electrical circuit can be drawn with ease.

Even those who have never used any analogy and never intend to do so, will be adversely affected by the fact that the old analogy has been used in making the definition of mechanical impedance. This unfortunate choice of definition will retard the development of vibration theory and make it all the more necessary to reply on electrical circuit theory by analogy.

The author nominates for oblivion the conventional left-handed mechanical-electrical analogy.

I wish to thank Mr. L. D. Montgomery and Mr. Earl Burns for their valuable suggestions.

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