Physics 404: Final Exam   Name (*print*): _____

"I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Dec. 20, 2013   *Sign* Honor Pledge: _____

Don't get bogged down on any problem you find difficult. Skip ahead and go through all 4 pages, then come back to it afterwards!! Note that not all later parts of problems depend on the answers to earlier parts, so you should read through entire problems. You may detach the equation sheet from the back of the exam for easier use.

1. Consider the p-*V* diagrams of two cyclic processes of a system starting at (V_A, p_A), expanding isothermally to (V_B, p_B), then either contracting isobarically to (V_C, p_B) OR decreasing in pressure isochorically to $(V_B, (D))$, and finally returning to initial point A adiabatically. Draw a sketch.

a) Which cyclic process, ABCA or ABDA, has larger W_{by}? (Explain briefly, but no calculation needed.)

b) Compare the heat energy *absorbed* by the two cycles. (Explain briefly, indicating on which leg[s] heat *enters* the system; no calculation is needed.)

c) Which is more efficient? (Explain briefly, but no calculation needed.)

d) Compare the total entropy change in going around the two cycles.

2. a) What is the average energy, in equilibrium at *T*, of a particle [in 1D] having energy $p^2/2m + kx^2$? (Note that *p* is momentum in this problem, not pressure!)

b) What is the average energy of a dimer like CO in equilibrium at *T* in 2D? (Exclude vibrations.)

c) What is the average energy, in equilibrium at *T*, of a particle [in 1D] having energy $p^2/2m + bx^8$?

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3. For each of the following statements about heat capacity C_V, indicate whether it is true (**T**) or false (**F**). If true, give a brief reason why. If false, give a counterexample.

a) At high temperature T, C_V is determined by equipartition.

b) C_V is an extensive quantity. (This is heat capacity, not specific heat.)

c) C_V is smaller than C_p. (No justification needed for this item)

d) Assuming a unique ground state, as $T \to 0$, $C_V \to 0$.

e) C_V increases monotonically with T; i.e. $dC_V/dT \geq 0$.

4. a) Circle those of the following systems for which $C \propto \exp(-\Delta/k_B T)$ as $T \to 0$. **b)** Next to those that you have circled, write what the energy gap Δ is in this case (in terms of variables, not numerical value).

Classical non-interacting particles (ideal gas)

Relativistic electrons

Rotating (but stationary) dimers (diatomic molecules)

Phonons in the Einstein model

Phonons in the Debye model

Paramagnets (2-state)

Ising model (square-lattice case)

Ising model, mean-field solution (square-lattice case)

5. Arrange the following characteristic temperatures from coldest to hottest:

T_c (Bose-Einstein condensation), T_D, T_F, T_λ (superfluid transition of ^{4}He), T_{neg}, T_{rand}, $T_{S=0}$

where T_{rand} is the temperature at which particles occupy energy levels randomly (regardless of their energy) and T_{neg} is a typical temperature for negative temperatures

6. a) Write down the integral that you would need to perform to compute the root mean squared speed associated with the Maxwell speed distribution in 3D; **b)** rewrite this integral in terms of an integral I over a dimensionless variable, extracting all the dependence on dimensional variables; **c)** what is v_{rms} in terms of dimensional variables and I?

7. Consider the depicted family of isotherms for the van der Waals (vdW) equation (cf. formula sheet).
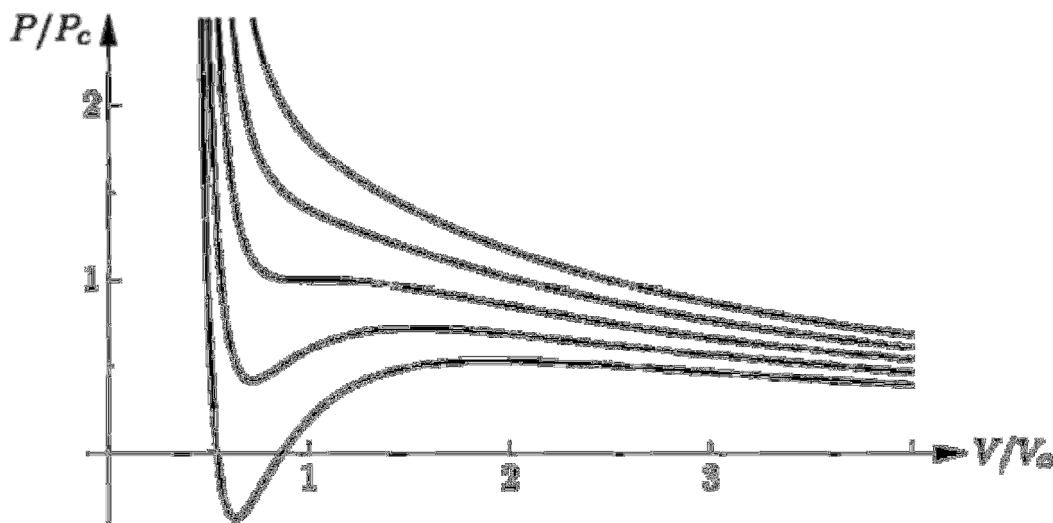
a) What are the values of $p'(V)$ and $p''(V)$ at the critical point? Draw an X at it in the plot.

b) Draw the Maxwell construction for the second-from-the-bottom isotherm. Indicate clearly the two values of the volumes of the coexisting phases; label them V_1 and $V_2 > V_1$.

c) For a higher T than in the last two parts (but still less than T_{cr}), how does $V_1(T)$ compare with V_1 in part b)? How does $V_2(T)$ compare with V_2? What happens to $V_1(T)$ and $V_2(T)$ as $T \rightarrow T_{cr}$?

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8. a) In an (intensive) electric field $\mathcal{E}$ and polarization p, we have $dH(S, \mathcal{E}, N) = T\,dS - p\,d\mathcal{E} + \mu\,dN$. Find the Maxwell relation for $\partial p / \partial S$.

b) Find the thermodynamic identity for the associated Gibbs free energy $G(T, \mathcal{E}, N)$.

c) Why does the relation $G = \mu N$ hold for G but not U, H, or F? (Show that $dH/dN \neq \mu$.)

9. Consider a system for which the heat capacity is $C_V(T) = A\,T^2$ for $0 < T < T_{max}$.

a) Find $S(T)$ for $0 < T < T_{max}$. You should invoke the 3rd law of thermodynamics.

b) Find $U(T)$ for $0 < T < T_{max}$.

c) Can T_{max} be ∞? Why or why not?

d) For the systems considered in this course, what is the likeliest dimensionality of this system?

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10. Recall the density of states in 3D for free electrons: $G(\varepsilon) = a\,N\,\varepsilon^{1/2}/\varepsilon_F^{3/2}$ where a is a number of order 1.

a) Find the value of $\langle \varepsilon^3 \rangle$ at $T = 0$ in terms of a, N, and ε_F).

b) From the Sommerfeld expansion on the formula sheet, show very quickly (a one-liner) that if $h(\varepsilon) = G(\varepsilon)\,H(\varepsilon)$, then

$$\int_0^\infty G(\varepsilon)\,H(\varepsilon)\,\overline{n}_{FD}(\varepsilon;T)\,d\varepsilon = \int_0^{\varepsilon_F} G(\varepsilon)\,H(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}\left(k_B T\right)^2 G\left(\varepsilon_F\right) H'\left(\varepsilon_F\right)$$

c) Find the leading thermal correction to $\langle \varepsilon^3 \rangle$ for small T (i.e. $T \ll T_F$).

d) Find the value of a.

e) In 1D would you expect $\langle \varepsilon^3 \rangle / N\varepsilon_F^3$ at $T = 0$ to be larger, smaller, or the same as in part a)?

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11. a) i) Sketch the chemical potential μ vs. T for a Bose gas. Label Tc clearly. **ii)** Also on the same graph, sketch N_0/N with a dashed line. **iii)** What is the value of N_0/N at $T/T_c = \frac{1}{4}$?

b) In 3D, below T_c, we find $N_{ex} = \dfrac{2}{\sqrt{\pi}} \dfrac{V}{\lambda_T^3} \displaystyle\int_0^\infty \dfrac{\sqrt{x}}{e^x - 1} dx$.

i) From what physical quantity does the $x^{1/2}$ come? (Circle one.)

   Density of states   Mean speed of bosons   Size of the box in which the bosons are confined

ii) What is the exponent of x in the 2D case? (If you do not know, at least say whether it increases, decreases, or stays at 1/2.)

iii) What is the exponent of x for harmonically trapped (confined) atoms in 3D? (If you do not know, at least say whether it increases, decreases, or stays at 1/2.)

iv) What is missing from the expression for N_{ex} that makes it invalid above T_c? (Note that the problem is not that N_{ex} can be replaced by N above T_c !)

v) Why do photons not undergo Bose-Einstein condensation?

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