

# Physics 402 Take Home Exam

## Due At 9:00 AM, Monday March 17, 2002

This exam is open notes and open book. You may also use Mathematica or other symbolic manipulation programs. **If you use Mathematica or a similar program you must include the output to get credit.** Do not seek outside help. (I trust you.) If you cannot do a section of the exam do not panic. The exam is written in such a way as you can often do a later section of a problem while missing earlier parts. To aid you in this, I will often ask you to show that something is true rather than asking for the answer. **To get credit you must show how you obtained your answer from the basic physical and mathematical principles.** You may use formulae that we derived in class or in the book as a starting point. Since you have considerable time on this exam, I fully expect your answers to be clear.

If you have questions you may e-mail me (cohen@physics.umd.edu) or call me at the office (301) 405-6117 or at home (301) 654-7702 (**Before 10:00 p.m.**)

- 1) Consider a general angular momentum operator  $\hat{J}$  which satisfies the usual commutation rules  $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$ . Define basis states  $|J, m\rangle$  in the usual way satisfying  $\hat{J}^2|J, m\rangle = \hbar^2 J(J+1)|J, m\rangle$  and  $\hat{J}_z|J, m\rangle = \hbar m|J, m\rangle$ . In this problem your task is to compute the matrix element  $\langle J, m' | \hat{J}_x^2 | J, m \rangle$ . If you do this properly you will find that the matrix element is zero unless  $m' = m$  or  $m' = m \pm 2$ . *Hint: You may wish to exploit the operators  $\hat{J}_+$  and  $\hat{J}_-$ .*
  
- 2) In class we studied the form for energy eigenstates for a particle moving in three dimensions in a central potential---i.e. a potential that only depended on the magnitude of  $r$  but which did not depend on angles. We found that the wave function for the energy eigenstates could be separated into the product of an angular function (a spherical harmonic) and a radial function, where the radial function was the solution of an eigenvalue equation in which there was an effective potential which depended on the index of the spherical harmonic. In this problem, I want you to do a similar analysis for a somewhat simpler problem --- a particle moving in two dimensions under the influence of a central potential. Thus, the time-independent Schrödinger equation will be given by: 
$$\left( \frac{-\hbar^2}{2M} \nabla^2 + V(r) \right) \psi(\vec{r}) = E \psi(\vec{r})$$
 with the Laplacian being two dimensional. It is sensible to work in polar coordinates. In polar coordinates the two dimensional Laplacian is given by 
$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$
. Assume that the equation is separable so that  $\psi(\vec{r}) = R(r)\Theta(\theta)$ .
  - a) Show that the separability of the equation implies that  $\frac{\partial^2}{\partial \theta^2} \Theta(\theta) = -m^2 \Theta(\theta)$  where  $m$  is a constant.

b) Solve for the equation in a) for  $\Theta(\theta)$ . Show on physical grounds that  $m$  must be an integer for these solutions to make sense.

c) Show that  $R(r)$  satisfies an eigenvalue equation of the following form

$$\left( \frac{-\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \right) + V_{\text{eff}}(r) \right) R(r) = E R(r) \text{ where } V_{\text{eff}}(r) \text{ depends on the original potential } V(r)$$

and on  $m$ . Find the form of  $V_{\text{eff}}(r)$ .

Note this problem is not totally artificial---in condensed matter physics one often encounters the situation where a particle is confined to move in a plane.

3) A hydrogen atom is in the state  $|\psi\rangle = \frac{1}{\sqrt{2}} |\psi_{1,0,0}\rangle + \frac{1}{2} |\psi_{2,1,1}\rangle + \frac{1}{2} |\psi_{2,0,0}\rangle$  where the state  $|\psi_{n,l,m}\rangle$  is an energy eigenstate characterized by the usual quantum numbers  $n$ ,  $l$ , and  $m$ .

a) Find the expectation value of the energy, the z component of the angular momentum and the square of the angular momentum in this state. That is find  $\langle \psi | \hat{H} | \psi \rangle$ ,  $\langle \psi | \hat{L}_z | \psi \rangle$  and  $\langle \psi | \hat{L}^2 | \psi \rangle$ .

b) What is the expectation value of the x-component of the angular momentum,  $\langle \psi | \hat{L}_x | \psi \rangle$ ?

c) If the z component of the angular momentum,  $L_z$ , is measured what is the probability the measured value will be  $+\hbar$ ? What is the probability that it will 0? What is the probability that it will  $-\hbar$ ? Briefly explain your reasoning.

4) The purpose of this problem is to consider what happens quantum mechanically to an atomic system when the nucleus at its center undergoes a nuclear decay. This has the effect of changing the atomic potential. The system we will consider is tritium (a form of heavy hydrogen with a nucleus composed of two neutrons and a proton) decaying via  $\beta$  decay into a form of Helium ( $^3\text{He}$ ) (which it does by the emission of an electron and an antineutrino). In this problem you may treat the nucleus as being very heavy compared to the electron and you may neglect all spin effects, relativity and so forth. This reduces the atomic physics problem to a one-particle quantum problem of an electron in a Coulomb potential. The nuclear emission process is very fast on the scale of atomic physics so that from the point of view of the atomic wave function it "looks like" the potential seen by the orbiting electron nearly instantly

changes from  $V(r) = -\frac{e^2}{(4\pi\epsilon_0)r}$  (the hydrogen potential) to  $V(r) = -\frac{2e^2}{(4\pi\epsilon_0)r}$  (the helium potential)

at the time of the decay. That is the problem essentially changes instantaneously from being that of the hydrogen atom to that of singly ionized helium (a single electron orbiting the helium nucleus).

a) As a preliminary we need to compute the wave function of singly ionized helium, which consists of a single electron orbiting a helium nucleus. This nucleus has a charge of  $+2e$  so  $V(r) = -\frac{2e^2}{(4\pi\epsilon_0)r}$ .

Assume the system is in its ground state---namely the state with  $n=1$ ,  $L=0$ ,  $m=0$ . Show that the helium atomic ground state has a wave function given by  $\psi_{\text{ground}}^{\text{He}}(\vec{r}) = \sqrt{8} \psi_{\text{ground}}^{\text{H}}(2\vec{r})$  where

$\psi_{\text{ground}}^H(\vec{r})$  is the ground state of the hydrogen wave function. *Hint: You do not need to redo the entire analysis we did for hydrogen wave function. You may exploit the results for hydrogen we derived in class and simply make the appropriate changes for a potential of different strength.*

- b) Assume that a tritium (heavy hydrogen) atom begins in its atomic ground state and subsequently undergoes  $\beta$  decay. The decay is essentially instantaneous on atomic physics scales so that immediately after the decay takes place the wave function is just the ground hydrogen wave function. Since after the decay the system is a singly ionized Helium atom, when one measures the energy of the atom one will find the system in an energy eigenstate of the singly ionized helium atom with some probability. What is the probability that it will be in the ground state of the singly ionized Helium atom? (You may use the wave function given in the previous section here whether or not you derived it. You may use Mathematica to evaluate any integrals you get).
- c) What is the expectation value of the energy of the atomic electron after the decay? *Hint: Recall that the energy expectation value is time independent if the Hamiltonian is fixed so that the energy expectation value for any time after the decay is identical to the expectation value immediately after the decay. The wave function immediately after the decay is known.*
- 5) In class we have studied the time evolution of a spin  $\frac{1}{2}$  particle in a magnetic field. We considered the case where the magnetic field was oriented along the z direction. This was convenient since we defined our basis states in terms of their  $m$  quantum number (which specifies the z-component of angular momentum). In this problem, I will ask you to do things *inconveniently*. Namely we will still work with our standard basis state  $|\uparrow\rangle$  and  $|\downarrow\rangle$  which correspond to spin up or down along the z axis, but this time the magnetic field is aligned in the x direction. Thus,
- 6)
- 7) the Hamiltonian is given by  $\hat{H} = -\gamma B_o \hat{s}_x$ . We may generally represent the state of the system as  $|\psi(t)\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle$  where  $|a(t)|^2 + |b(t)|^2 = 1$ .
- a) Show that the time evolution for the coefficients  $a(t)$  and  $b(t)$  is given by the following equation:
- $$-\frac{1}{2}\gamma B_o \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i \frac{d}{dt} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$
- b) Verify mathematically that the solution to the equation in a) is given by
- $$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = c_1 e^{i\gamma B_o t/2} \begin{pmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{pmatrix} + c_2 e^{-i\gamma B_o t/2} \begin{pmatrix} \sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix}$$
- where  $c_1$  and  $c_2$  are constants fixed by the initial conditions. Explain briefly the physical significance of the two terms in terms of eigenvectors of operators relevant to the problem.
- c) Find expressions for the expectation values  $\langle \hat{s}_x \rangle$ ,  $\langle \hat{s}_y \rangle$ ,  $\langle \hat{s}_z \rangle$  as functions of time expressed in terms of  $a(0)$ ,  $b(0)$  the initial values of  $a(t)$  and  $b(t)$ . If done correctly  $\langle \hat{s}_x \rangle$  should be independent of time. Verify that this is true and briefly comment on why it is expected.