PROBLEM 5.15 (a) $\psi = A\sin(kx) + B\cos(kx)$; $A\sin(ka) = \left[e^{iKa} - \cos(ka)\right]B \Rightarrow$

$$\psi = A\sin(kx) + \frac{A\sin(ka)}{\left[e^{iKa} - \cos(ka)\right]}\cos kx = \frac{A}{\left[e^{iKa} - \cos(ka)\right]}\left\{e^{iKa}\sin(kx) - \sin(kx)\cos(ka) + \cos(kx)\sin(ka)\right\}$$

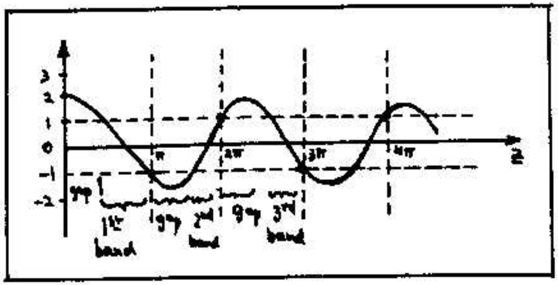
$$= C\left\{\sin(kx) + e^{-iKa}\sin\left[k(a-x)\right]\right\}, \text{ where } C \equiv \frac{Ae^{iKa}}{e^{iKa} - \cos(ka)}.$$

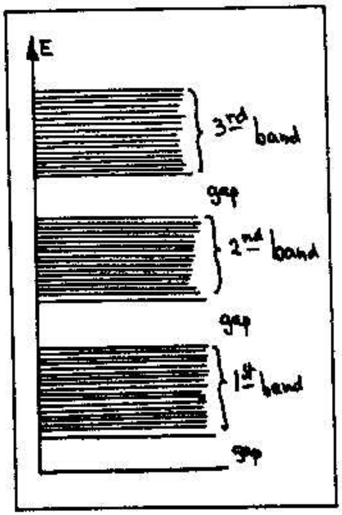
(b) If $ka = Ka = j\pi$, then $[5.63] \Rightarrow A\sin(j\pi) = \left[e^{ij\pi} - \cos(j\pi)\right]B \Rightarrow 0 = \left[(-1)^j - (-1)^j\right]B = 0$ — no condition on A or B. In this case [5.61] holds automatically, and [5.62] gives

$$kA - (-1)^j k\left[A(-1)^j - 0\right] = -\frac{2m\alpha}{\hbar^2}B \Rightarrow \underline{B=0}. \text{ So } \boxed{\psi = A\sin(kx)}.$$

ψ is zero at each delta function, so the wave function never "feels" the potential at all.

Problem 5.17





There are N allowed energies in every band — including the first.

The energy at the top of the j^{th} band is given by $z = j\pi$, or

$$E_j = \frac{\hbar^2 k^2 a^2}{2ma^2} = \frac{\hbar^2 z^2}{2ma^2} = \boxed{\frac{\hbar^2 j^2 \pi^2}{2ma^2}}.$$

Problem 5.18 $K = \frac{2\pi n}{Na} \Rightarrow Ka = 2\pi \frac{n}{N}$; $n = 0, 1, 2, \dots, N-1$ (eq. [5.56] and page 202).

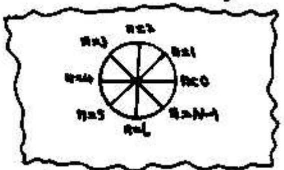
$\underline{N=1} \Rightarrow n=0 \Rightarrow \cos(Ka) = 1$. Nondegenerate.

$\underline{N=2} \Rightarrow n=0,1 \Rightarrow \cos(Ka) = 1, -1$. Nondegenerate.

$\underline{N=3} \Rightarrow n=0,1,2 \Rightarrow \cos(Ka) = 1, -\frac{1}{2}, -\frac{1}{2}$. First $\Rightarrow$ nondegenerate, other two $\Rightarrow$ degenerate.

$\underline{N=4} \Rightarrow n=0,1,2,3 \Rightarrow \cos(Ka) = 1, 0, -1, 0$. First and third $\Rightarrow$ nondegenerate, other two $\Rightarrow$ degenerate.

Evidently they are doubly degenerate (two different n's give same $\cos(Ka)$) <u>except</u> when $\cos(Ka) = \pm 1$ — i.e. at the $\boxed{\text{top or bottom of a band}}$. The point is that the Bloch factors e^{iKa} lie at equal angles in the complex plane, starting with 1 (see figure); by symmetry, there is always one with negative imaginary part symmetrically opposite each one with positive imaginary part — and the two have the same <u>real</u> part ($\cos(Ka)$). Only points which fall <u>on</u> the real axis have no twins.



(example for N = 8)

CHAPTER 6

PROBLEM 6.1 $\psi_n^0(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$, so $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \frac{2}{a}\alpha \int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\delta(x - a/2)\,dx$.

$E_n^1 = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right) = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{2}\right) = \boxed{\begin{cases} 0, & \text{if } n \text{ is even,} \\ 2\alpha/a, & \text{if } n \text{ is odd.} \end{cases}}$ For even n the wave function is zero at the location of the perturbation $(x = a/2)$, so it never "feels" H'.

PROBLEM 6.2 (a) $E_n = (n+\frac{1}{2})\hbar\omega'$, where $\omega' = \sqrt{k(1+\epsilon)/m} = \omega\sqrt{1+\epsilon} = \omega\left(1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 \cdots\right)$.

$\therefore \boxed{E_n = (n+\tfrac{1}{2})\hbar\omega\sqrt{1+\epsilon} = (n+\tfrac{1}{2})\hbar\omega\left(1 + \tfrac{\epsilon}{2} - \tfrac{1}{8}\epsilon^2 + \cdots\right).}$

(b) $H' = \frac{1}{2}k'x^2 - \frac{1}{2}kx^2 = \frac{1}{2}kx^2(1+\epsilon-1) = \epsilon\left(\frac{1}{2}kx^2\right) = \epsilon V$, where V is the unperturbed potential energy. So $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \epsilon\langle n|V|n\rangle$, with $\langle n|V|n\rangle$ the expectation value of the (unperturbed) potential energy in the n^{th} unperturbed state. This is most easily obtained from the Virial theorem (Problem 3.53), but it can also be derived algebraically (Problem 2.37). In this case the Virial Theorem says $\langle T\rangle = \langle V\rangle$. But $\langle T\rangle + \langle V\rangle = E_n$. So $\langle V\rangle = \frac{1}{2}E_n^0 = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$. So $\boxed{E_n^1 = \frac{\epsilon}{2}(n+\frac{1}{2})\hbar\omega}$, which is precisely the ϵ^1 term in the power series from part (a).