CHAPTER 5

PROBLEM 5.1 (a) $(m_1+m_2)\vec R = m_1\vec r_1 + m_2\vec r_2 = m_1\vec r_1 + m_2(\vec r_1 - \vec r_2) = (m_1+m_2)\vec r_1 - m_2\vec r_2 \Rightarrow$

$$\vec r_1 = \vec R + \frac{m_2}{m_1+m_2}\vec r = \vec R + \frac{\mu}{m_1}\vec r. \checkmark$$

$(m_1+m_2)\vec R = m_1(\vec r_2 + \vec r) + m_2\vec r_2 = (m_1+m_2)\vec r_2 + m_1\vec r_1 \Rightarrow \vec r_2 = \vec R - \frac{m_1}{m_1+m_2}\vec r = \vec R - \frac{\mu}{m_2}\vec r. \checkmark$

Let $\vec R = (X,Y,Z)$, $\vec r = (x,y,z)$. $(\nabla_1)_x = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \left(\frac{m_1}{m_1+m_2}\right)\frac{\partial}{\partial X} + (1)\frac{\partial}{\partial x} = \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x$

$$\text{So } \vec\nabla_1 = \frac{\mu}{m_2}\vec\nabla_R + \vec\nabla_r \checkmark$$

$$(\nabla_2)_x = \frac{\partial}{\partial x_2} = \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} = \left(\frac{m_2}{m_1+m_2}\right)\frac{\partial}{\partial X} - (1)\frac{\partial}{\partial x} = \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x \Rightarrow \vec\nabla_2 = \frac{\mu}{m_1}\vec\nabla_R - \vec\nabla_r. \checkmark$$

(b) $\nabla_1^2\psi = \vec\nabla_1\cdot(\vec\nabla_1\psi) = \vec\nabla_1\cdot\left[\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right] = \frac{\mu}{m_2}\vec\nabla_R\cdot\left(\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right) + \vec\nabla_r\cdot\left(\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right)$

$= \left(\frac{\mu}{m_2}\right)^2\nabla_R^2\psi + 2\frac{\mu}{m_2}\left(\vec\nabla_r\cdot\vec\nabla_R\right)\psi + \nabla_r^2\psi$. Likewise $\nabla_2^2\psi = \left(\frac{\mu}{m_1}\right)^2\nabla_R^2\psi - 2\frac{\mu}{m_1}\left(\vec\nabla_r\cdot\vec\nabla_R\right)\psi + \nabla_r^2\psi$.

$\therefore H\psi = -\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(\vec r_1,\vec r_2)\psi = -\frac{\hbar^2}{2}\left\{\frac{\mu^2}{m_1m_2^2}\nabla_R^2 + \frac{2\mu}{m_1m_2}\vec\nabla_r\cdot\vec\nabla_R + \frac{1}{m_1}\nabla_r^2 + \frac{\mu^2}{m_2m_1^2}\nabla_R^2 - \frac{2\mu}{m_2m_1}\vec\nabla_r\cdot\vec\nabla_R + \frac{1}{m_2}\nabla_r^2\right\}\psi$

$+V(\vec r)\psi = -\frac{\hbar^2}{2}\left[\frac{\mu^2}{m_1m_2}\left(\frac{1}{m_2}+\frac{1}{m_1}\right)\nabla_R^2 + \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla_r^2\right]\psi + V(\vec r)\psi = E\psi$. But $\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{m_1+m_2}{m_1m_2} = \frac{1}{\mu}$, so

$$\frac{\mu^2}{m_1m_2}\left(\frac{1}{m_2}+\frac{1}{m_1}\right) = \frac{\mu}{m_1m_2} = \frac{m_1m_2}{m_1m_2(m_1+m_2)} = \frac{1}{m_1+m_2}. \quad\therefore\quad -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec r)\psi = E\psi. \checkmark$$

(c) Put in $\psi = \psi_r(\vec r)\psi_R(\vec R)$, and divide by $\psi_r\,\psi_R$:

$\left[-\frac{\hbar^2}{2(m_1+m_2)}\frac{1}{\psi_R}\nabla_R^2\psi_R\right] + \left[-\frac{\hbar^2}{2\mu}\frac{1}{\psi_r}\nabla_r^2\psi_r + V(\vec r)\right] = E$. First term depends only on $\vec R$, second only on $\vec r$,

so each must be a constant — call them E_R and E_r, respectively. Then:

$$\boxed{-\frac{\hbar^2}{2(m_1+m_2)}\nabla^2\psi_R = E_R\psi_R}\;;\quad \boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi_r + V(\vec r)\psi_r = E_r\psi_r},\ \text{with } \boxed{E_R + E_r = E}.$$

PROBLEM 5.2 (a) From [4.77], E_1 is proportional to mass, so $\frac{\Delta E_1}{E_1} = \frac{\Delta m}{\mu} = \frac{m-\mu}{\mu} = \frac{m(m+M)}{mM} - \frac{M}{M} = \frac{m}{M}$.

So the fractional error is the ratio of the electron mass to the proton mass: $\frac{9.109\times10^{-31}\,\text{kg}}{1.673\times10^{-27}\,\text{kg}} = 5.44\times10^{-4}$.

The percent error is $\boxed{0.054\%}$ (pretty small).

(b) From [4.94], R is proportional to m, so $\frac{\Delta(1/\lambda)}{(1/\lambda)} = \frac{\Delta R}{R} = \frac{\Delta\mu}{\mu} = \frac{-(1/\lambda^2)\Delta\lambda}{(1/\lambda)} = -\frac{\Delta\lambda}{\lambda}$.

So (in magnitude) $\frac{\Delta\lambda}{\lambda} = \frac{\Delta\mu}{\mu}$. $\mu = \frac{mM}{m+M}$, where m = electron mass, and M = nuclear mass.

$$\Delta\mu = \frac{m\,2m_p}{m+2m_p} - \frac{m\,m_p}{m+m_p} = \frac{m\,m_p}{(m+m_p)(m+2m_p)}\left[2m + 2m_p - m - 2m_p\right] = \frac{m^2 m_p}{(m+m_p)(m+2m_p)} = \frac{m\mu}{m+2m_p}.$$

$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\mu}{\mu} = \frac{m}{m+2m_p} \cong \frac{m}{2m_p}$, so $\boxed{\Delta\lambda = \frac{m}{2m_p}\lambda_h}$, where λ_h is the hydrogen wavelength:

$$\frac{1}{\lambda} = R\left(\frac14 - \frac19\right) = \frac{5}{36}R \Rightarrow \lambda = \frac{36}{5R} = \frac{36}{5(1.097\times10^7)}\,\text{m} = 6.563\times10^{-7}\,\text{m}. \quad \Delta\lambda = \frac{9.109\times10^{-31}}{2(1.673\times10^{-27})}(6.563\times10^{-7})\,\text{m}$$

$$= \boxed{1.79\times10^{-10}\,\text{m}}$$

(c) $\mu = \frac{mm}{m+m} = \frac{m}{2}$, so the energy is half what it would be for hydrogen: $\frac{13.6\text{eV}}{2} = \boxed{6.8\text{eV}}$.

(d) $\mu = \frac{m_p m_\mu}{m_p + m_\mu}$; $R \propto \mu$, so R is changed by a factor $\frac{m_p m_\mu}{m_p+m_\mu} \cdot \frac{m_p+m_e}{m_p m_e} = \frac{m_\mu(m_p+m_e)}{m_e(m_p+m_\mu)}$, as compared with hydrogen. For hydrogen, $\frac{1}{\lambda} = R\left(1-\frac{1}{4}\right) = \frac{3}{4}R \Rightarrow \lambda = \frac{4}{3R} = \frac{4}{3(1.097\times10^{7})}\text{m} = 1.215\times10^{-7}\text{m}$, and $\lambda \propto \frac{1}{R}$, so for muonic hydrogen the Lyman-alpha line is at

$$\lambda = \frac{m_e(m_p+m_\mu)}{m_\mu(m_p+m_e)}(1.215\times10^{-7}\text{m}) = \frac{1}{206.77}\frac{(1.673\times10^{-27} + 206.77\times9.109\times10^{-31})}{(1.673\times10^{-27} + 9.109\times10^{-31})}(1.215\times10^{-7}\text{m}) =$$

$$\boxed{6.54\times10^{-10}\text{m}}$$

PROBLEM 5.3 (a) $1 = \int|\psi_\pm|^2 d^3r_1 d^3r_2 = |A|^2\int[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)]^*[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)]d^3r_1 d^3r_2$

$$= |A|^2\Big\{\int|\psi_a(r_1)|^2 d^3r_1\int|\psi_b(r_2)|^2 d^3r_2 \pm \int\psi_a(r_1)^*\psi_b(r_1)d^3r_1\int\psi_b(r_2)^*\psi_a(r_2)d^3r_2 \pm \int\psi_b(r_1)^*\psi_a(r_1)d^3r_1\int\psi_a(r_2)^*\psi_b(r_2)d^3r_2$$

$$+ \int|\psi_b(r_1)|^2 d^3r_1\int|\psi_a(r_2)|^2 d^3r_2\Big\} = |A|^2(1\cdot1 \pm 0\cdot0 \pm 0\cdot0 + 1\cdot1) = 2|A|^2 \Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}.$$

(b) $1 = |A|^2\int[2\psi_a(r_1)\psi_a(r_2)]^*[2\psi_a(r_1)\psi_a(r_2)]d^3r_1 d^3r_2 = 4|A|^2\int|\psi_a(r_1)|^2 d^3r_1\int|\psi_a(r_2)|^2 d^3r_2 = 4|A|^2$. $\boxed{A = \frac{1}{2}}$.

PROBLEM 5.4 (a) $\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2\psi}{dx_2^2} = E\psi}$ (for $0 \le x_1, x_2 \le a$ — otherwise $\psi = 0$).

$$\psi = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] \Rightarrow \frac{d^2\psi}{dx_1^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \left(-\frac{2\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_2^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \left(-\frac{\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], \text{ so } \left(\frac{d^2\psi}{dx_1^2} + \frac{d^2\psi}{dx_2^2}\right) = -\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{a}\right)^2\right]\psi = -5\frac{\pi^2}{a^2}\psi.$$

$$\therefore -\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dx_1^2} + \frac{d^2\psi}{dx_2^2}\right) = \frac{5\hbar^2}{2m}\frac{\pi^2}{a^2}\psi = E\psi, \text{ with } E = \frac{5\pi^2\hbar^2}{2ma^2} = 5K. \checkmark$$

(b) Distinguishable: $\boxed{\psi_{22} = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right), \text{ with } E_{22} = 8K}$ (nondegenerate)

$\boxed{\begin{matrix}\psi_{13} = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) \\ \psi_{31} = \frac{2}{a}\sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\end{matrix}\Bigg\}, \text{ with } E_{13} = E_{31} = 10K}$ (doubly degenerate)

Identical Bosons: $\boxed{\psi_{22} = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right), E_{22} = 8K}$ (nondegenerate)

$\boxed{\psi_{13} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) + \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], E_{13} = 10K}$ (nondegenerate)

Identical Fermions: $\boxed{\psi_{13} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], E_{13} = 10K}$ (nondegenerate)

$\boxed{\psi_{23} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right], E_{23} = 13K}$ (nondegenerate)

PROBLEM 5.5 (a) [5.19], with $\langle x\rangle_n = \frac{a}{2}$ (Problem 2.5) and $\langle x^2\rangle_n = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$ (Problem 2.5) $\Rightarrow$

$$\langle(x_1 - x_2)^2\rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2m^2\pi^2}\right) - 2\cdot\frac{a}{2}\cdot\frac{a}{2} = \boxed{a^2\left\{\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right\}}.$$