PROBLEM 4.32 There are 3 states: $\chi_+ = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\chi_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\0\\1\end{pmatrix}$. $S_z\chi_+ = \hbar\chi_+$, $S_z\chi_0 = 0$, $S_z\chi_- = -\hbar\chi_-$.

So $\boxed{S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}}$ $\left.\begin{matrix} S_+\chi_+ = 0,\ S_+\chi_0 = \hbar\sqrt{2}\,\chi_+,\ S_+\chi_- = \hbar\sqrt{2}\,\chi_0 \\ S_-\chi_+ = \hbar\sqrt{2}\,\chi_0,\ S_-\chi_0 = \hbar\sqrt{2}\,\chi_-,\ S_-\chi_- = 0 \end{matrix}\right\}$ from [4.136].

$$\therefore\ S_+ = \sqrt{2}\hbar\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad S_- = \sqrt{2}\hbar\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}.\quad S_x = \frac{1}{2}(S_+ + S_-) = \boxed{\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}}.$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \boxed{\frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}}$$

PROBLEM 4.33 (a) Using [4.151] and [4.163]:

$$c_+^{(x)} = \chi_+^{(x)\dagger}\chi = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right].$$

$$P_+^{(x)}(t) = |c_+^{(x)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$$

$$= \frac{1}{2}\left\{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}\right)\right\} = \frac{1}{2}\left(1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos(\gamma B_0 t)\right)$$

$$= \boxed{\frac{1}{2}\left(1 + \sin\alpha\cos(\gamma B_0 t)\right)}.$$

(b) From Problem 4.30 (a): $\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$. $c_+^{(y)} = \chi_+^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}(1\ -i)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix}$

$c_+^{(y)} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$;

$P_+^{(y)}(t) = |c_+^{(y)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + i\sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$

$= \frac{1}{2}\left\{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} - e^{-i\gamma B_0 t}\right)\right\} = \frac{1}{2}\left(1 - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin(\gamma B_0 t)\right)$

$= \boxed{\frac{1}{2}\left(1 - \sin\alpha\sin(\gamma B_0 t)\right)}$.

(c) $\chi_+^{(z)} = \begin{pmatrix}1\\ 0\end{pmatrix}$. $c_+^{(z)} = (1\ 0)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}$. $P_+^{(z)}(t) = |c_+^{(z)}|^2 = \boxed{\cos^2\frac{\alpha}{2}}$.

Problem 4.34 (a) $H = -\gamma\vec{B}\cdot\vec{S} = -\gamma B_0\cos\omega t\, S_z = \boxed{-\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$.

(b) $\chi(t) = \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$, with $\alpha(0) = \beta(0) = 1/\sqrt{2}$.

$i\hbar\frac{\partial\chi}{\partial t} = i\hbar\begin{pmatrix}\dot\alpha\\ \dot\beta\end{pmatrix} = H\chi = -\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$.

$\dot\alpha = i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\,\alpha \Rightarrow \frac{d\alpha}{\alpha} = i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\, dt \Rightarrow \ln\alpha = \frac{i\gamma B_0}{2}\frac{\sin\omega t}{\omega} + \text{constant}.$

$\alpha(t) = Ae^{i(\gamma B_0/2\omega)\sin\omega t}$. $\alpha(0) = A = \frac{1}{\sqrt{2}}$, so $\alpha(t) = \frac{1}{\sqrt{2}}e^{i(\gamma B_0/2\omega)\sin\omega t}$.

$\dot\beta = -i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\,\beta \Rightarrow \beta(t) = \frac{1}{\sqrt{2}}e^{-i(\gamma B_0/2\omega)\sin\omega t}$.

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i(\gamma B_0/2\omega)\sin\omega t}\\ e^{-i(\gamma B_0/2\omega)\sin\omega t}\end{pmatrix}}.$$

(c) $c_-^{(x)} = \chi_-^{(x)\dagger}\chi = \frac{1}{2}(1\ -1)\begin{pmatrix}e^{i(\gamma B_0/2\omega)\sin\omega t}\\ e^{-i(\gamma B_0/2\omega)\sin\omega t}\end{pmatrix} = \frac{1}{2}\left[e^{i(\gamma B_0/2\omega)\sin\omega t} - e^{-i(\gamma B_0/2\omega)\sin\omega t}\right]$

$= i\sin\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]$. $P_-^{(x)}(t) = |c_-^{(x)}|^2 = \boxed{\sin^2\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]}$.

(d) The argument of $\sin^2$ must reach $\pi/2$ (so $P = 1$). $\therefore\ \frac{\gamma B_0}{2\omega} = \frac{\pi}{2}$, or $\boxed{B_0 = \frac{\pi\omega}{\gamma}}$.

Problem 4.35 (a) $S_-|10\rangle = (S_-^{(1)} + S_-^{(2)})\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left[(S_-\uparrow)\downarrow + (S_-\downarrow)\uparrow + \uparrow(S_-\downarrow) + \downarrow(S_-\uparrow)\right]$.

But $S_-\uparrow = \hbar\downarrow$, $S_-\downarrow = 0$ (eq. [4.143]), so $S_-|10\rangle = \frac{1}{\sqrt{2}}\left[\hbar\downarrow\downarrow + 0 + 0 + \hbar\downarrow\downarrow\right] = \sqrt{2}\hbar\downarrow\downarrow = \sqrt{2}\hbar|1\,-1\rangle$.

(b) $S_\pm|00\rangle = (S_\pm^{(1)} + S_\pm^{(2)})\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left[(S_\pm\uparrow)\downarrow - (S_\pm\downarrow)\uparrow + \uparrow(S_\pm\downarrow) - \downarrow(S_\pm\uparrow)\right]$

$S_+|00\rangle = \frac{1}{\sqrt{2}}\left(0 - \hbar\uparrow\uparrow + \hbar\uparrow\uparrow - 0\right) = 0$; $S_-|00\rangle = \frac{1}{\sqrt{2}}\left(\hbar\downarrow\downarrow - 0 + 0 - \hbar\downarrow\downarrow\right) = 0$. ✓

(c) $S^2|11\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}\right]\uparrow\uparrow = (S^2\uparrow)\uparrow + \uparrow(S^2\uparrow) + 2\left[(S_x\uparrow)(S_x\uparrow) + (S_y\uparrow)(S_y\uparrow) + (S_z\uparrow)(S_z\uparrow)\right]$

$= \frac{3}{4}\hbar^2\uparrow\uparrow + \frac{3}{4}\hbar^2\uparrow\uparrow + 2\left[\frac{\hbar}{2}\downarrow\frac{\hbar}{2}\downarrow + \frac{i\hbar}{2}\downarrow\frac{i\hbar}{2}\downarrow + \frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow\right] = \frac{3}{2}\hbar^2\uparrow\uparrow + 2\left[\frac{\hbar^2}{4}\uparrow\uparrow\right] = 2\hbar^2\uparrow\uparrow$

$= 2\hbar^2|11\rangle = (1)(1+1)\hbar^2|11\rangle$, as it should be.

$$S^2|1\,-1\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\,\vec{S}^{(1)}\cdot\vec{S}^{(2)}\right]\downarrow\downarrow = \frac{3\hbar^2}{4}\downarrow\downarrow + 3\frac{\hbar^2}{4}\downarrow\downarrow + 2\left[(S_x\downarrow)(S_x\downarrow) + (S_y\downarrow)(S_y\downarrow) + (S_z\downarrow)(S_z\downarrow)\right]$$

$$= \frac{3}{2}\hbar^2\downarrow\downarrow + 2\left[\left(\frac{\hbar}{2}\uparrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(-\frac{i\hbar}{2}\uparrow\right)\left(-\frac{i\hbar}{2}\uparrow\right) + \left(-\frac{\hbar}{2}\downarrow\right)\left(-\frac{\hbar}{2}\downarrow\right)\right] = \frac{3}{2}\hbar^2\downarrow\downarrow + 2\,\frac{\hbar^2}{4}\downarrow\downarrow = 2\hbar^2\downarrow\downarrow = 2\hbar^2|1\,-1\rangle. \checkmark$$

PROBLEM 4.36 (a) $\frac{1}{2}$ and $\frac{1}{2}$ gives 1 or zero; $\frac{1}{2}$ and 1 gives $3/2$ or $1/2$, $\frac{1}{2}$ and 0 gives $1/2$ only.
So baryons can have $\boxed{\text{spin } 3/2 \text{ or spin } 1/2}$ (and the latter can be achieved in two distinct ways).
[Incidentally, the lightest baryons do carry spin $1/2$ (proton, neutron, etc) or $3/2$ (Δ, Ω^-, etc) — heavier baryons can have higher total spin, but this is because the quarks have orbital angular momentum as well.]

(b) $\frac{1}{2}$ and $\frac{1}{2}$ gives $\boxed{\text{one or zero}}$. [Again, these are the observed spins for the lightest mesons — π's and K's have spin 0, ρ's and ω's have spin 1.]
