PROBLEM 4.27

(a) $[S_x, S_y] = S_xS_y - S_yS_x = \frac{\hbar^2}{4}\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} = \frac{\hbar^2}{4}\left\{\begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix}\right\}$

$= \frac{\hbar^2}{4}\begin{pmatrix}2i&0\\0&-2i\end{pmatrix} = i\hbar\,\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\hbar S_z$. ✓

(b) $\sigma_x\sigma_x = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1 = \sigma_y\sigma_y = \sigma_z\sigma_z$, so $\sigma_j\sigma_j = 1$ for $j = x, y,$ or z.

$\sigma_x\sigma_y = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = i\sigma_z$; $\sigma_y\sigma_z = \begin{pmatrix}0&i\\i&0\end{pmatrix} = i\sigma_x$; $\sigma_z\sigma_x = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = i\sigma_y$; similarly $\sigma_y\sigma_x = \begin{pmatrix}-i&0\\0&i\end{pmatrix} = -i\sigma_z$;

$\sigma_z\sigma_y = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\,\sigma_x$; $\sigma_x\sigma_z = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\sigma_y$. Equation [4.153] puts all this into a single equation. [Note: $\epsilon_{jkl} = \begin{cases} 1 \text{ if } jkl = xyz, yzx, \text{ or } zxy \\ -1 \text{ if } jkl = xzy, yxz, \text{ or } zyx \end{cases}$ and zero otherwise.]

Problem 4.28 (a) $\chi^\dagger\chi = |A|^2(9+16) = 25\,|A|^2 = 1 \rightarrow \boxed{A = 1/5}$.

(b) $\langle S_x\rangle = \chi^\dagger S_x\chi = \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}4\\ 3i\end{pmatrix} = \frac{\hbar}{50}(-12i+12i) = \boxed{0}$.

$\langle S_y\rangle = \chi^\dagger S_y\chi = \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}-4i\\ -3\end{pmatrix} = \frac{\hbar}{50}(-12-12) = -\frac{24}{50}\hbar = \boxed{-\frac{12}{25}\hbar}$.

$\langle S_z\rangle = \chi^\dagger S_z\chi = \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\ -4\end{pmatrix} = \frac{\hbar}{50}(9-16) = \boxed{-\frac{7}{50}\hbar}$.

(c) $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}$ (always, for spin 1/2), so $\sigma_{S_x}^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0$, so $\boxed{\sigma_{S_x} = \frac{\hbar}{2}}$.

$\sigma_{S_y}^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-576) = \frac{49}{2500}\hbar^2 \Rightarrow \boxed{\sigma_{S_y} = \frac{7}{50}\hbar}$.

$\sigma_{S_z}^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-49) = \frac{576}{2500}\hbar^2 \Rightarrow \boxed{\sigma_{S_z} = \frac{12}{25}\hbar}$.

(d) $\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_z\rangle| = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar$; ✓ (right at the uncertainty limit).

$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar\cdot\frac{12}{25}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_x\rangle| = 0$; ✓ (trivial).

$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar\cdot\frac{\hbar}{2} \overset{?}{\geq} \frac{\hbar}{2}|\langle S_y\rangle| = \frac{\hbar}{2}\cdot\frac{12}{25}\hbar$; ✓ (right at the uncertainty limit).

Problem 4.29 $\langle S_x\rangle = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}b\\ a\end{pmatrix} = \boxed{\frac{\hbar}{2}(a^*b + b^*a)} = \hbar\,\mathrm{Re}(ab^*)$.

$\langle S_y\rangle = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}-ib\\ ia\end{pmatrix} = \frac{\hbar}{2}(-ia^*b + iab^*) = \boxed{\frac{\hbar}{2}i(ab^* - a^*b)} = -\hbar\,\mathrm{Im}(ab^*)$.

$\langle S_z\rangle = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}a\\ -b\end{pmatrix} = \frac{\hbar}{2}(a^*a - b^*b) = \boxed{\frac{\hbar}{2}(|a|^2 - |b|^2)}$.

$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{\hbar^2}{4}$; $S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}$; $S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}$; so

$\boxed{\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \hbar^2/4}$. $\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{3}{4}\hbar^2 \overset{?}{=} s(s+1)\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 = \langle S^2\rangle$. ✓

Problem 4.30 (a) $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. $\det\begin{pmatrix}-\lambda & -i\hbar/2\\ i\hbar/2 & -\lambda\end{pmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$ (of course).

$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow -i\beta = \pm\alpha$; $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha|^2 + |\alpha|^2 = 1 \Rightarrow \alpha = 1/\sqrt{2}$.

$\boxed{\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix};\ \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}}$.

(b) $c_+ = \chi_+^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib)$. $\boxed{+\frac{\hbar}{2}\text{, with probability } \frac{1}{2}|a-ib|^2}$.

$c_- = \chi_-^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}(a + ib)$. $\boxed{-\frac{\hbar}{2}\text{, with probability } \frac{1}{2}|a+ib|^2}$.

$P_+ + P_- = \frac{1}{2}\left[(a^* + ib^*)(a - ib) + (a^* - ib^*)(a + ib)\right] = \frac{1}{2}\left[|a|^2 - ia^*b + iab^* + |b|^2 + |a|^2 + ia^*b - iab^* + |b|^2\right]$
$= |a|^2 + |b|^2 = 1.$ ✓ (c) $\boxed{\hbar^2/4}$, probability $\boxed{1}$.

Problem 4.31 $S_r = \vec{S}\cdot\hat{r} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta$

$$= \frac{\hbar}{2}\left\{\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0\end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0\end{pmatrix} + \begin{pmatrix}\cos\theta & 0 \\ 0 & -\cos\theta\end{pmatrix}\right\} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & (\sin\theta)(\cos\phi - i\sin\phi) \\ (\sin\theta)(\cos\phi + i\sin\phi) & -\cos\theta\end{pmatrix}$$

$$= \boxed{\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}}. \quad \det\begin{pmatrix}(\frac{\hbar}{2}\cos\theta - \lambda) & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & (-\frac{\hbar}{2}\cos\theta - \lambda)\end{pmatrix} = -\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0 \Rightarrow$$

$$\lambda^2 = \frac{\hbar^2}{4}(\sin^2\theta + \cos^2\theta) = \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\hbar/2} \text{ (of course).}$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \Rightarrow \alpha\cos\theta + \beta e^{-i\phi}\sin\theta = \pm\alpha;\ \beta = e^{i\phi}\frac{(\pm 1 - \cos\theta)}{\sin\theta}\alpha.$$

Upper sign: use $1 - \cos\theta = 2\sin^2\theta/2$, $\sin\theta = 2\sin\theta/2\cos\theta/2$: $\beta = e^{i\phi}\frac{\sin\theta/2}{\cos\theta/2}\alpha$.

$$|\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha^2| + \frac{\sin^2\theta/2}{\cos^2\theta/2}|\alpha|^2 = |\alpha|^2\frac{1}{\cos^2\theta/2} = 1 \Rightarrow \alpha = \cos\theta/2;\ \beta = e^{i\phi}\sin\theta/2.$$

$$\therefore \boxed{\chi_+^{(r)} = \begin{pmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{pmatrix}}.$$

Lower sign: Use $1 + \cos\theta = 2\cos^2\theta/2$: $\beta = -e^{i\phi}\frac{\cos\theta/2}{\sin\theta/2}\alpha$; $1 = |\alpha|^2\left(1 + \frac{\cos^2\theta/2}{\sin^2\theta/2}\right) = |\alpha|^2\frac{1}{\sin^2\theta/2}$.

$\therefore \alpha = \sin\theta/2$, $\beta = -e^{i\phi}\cos\theta/2$. $$\boxed{\chi_-^{(r)} = \begin{pmatrix}\sin\theta/2 \\ -e^{i\phi}\cos\theta/2\end{pmatrix}}.$$