R_{32}. $(n=3, l=2)$. $a_1 = \frac{\;}{(1)(6)} a_0 = 0$, etc.

Problem 4.11 (a) [4.31] $\Rightarrow \int_0^\infty |R|^2 r^2\, dr = 1$. [4.82] $\Rightarrow R_{20} = \left(\frac{a_0}{2a}\right)\left(1-\frac{r}{2a}\right)e^{-r/2a}$. Let $\frac{r}{a} \equiv z$.

$$1 = \left(\frac{a_0}{2a}\right)^2 a^3 \int_0^\infty \left(1-\frac{z}{2}\right)^2 e^{-z} z^2\, dz = \frac{a_0^2 a}{4}\int_0^\infty \left(z^2 - z^3 + \frac{1}{4} z^4\right) e^{-z}\, dz = \frac{a_0^2 a}{4}\left[2-6+\frac{1}{4}\cdot 24\right] = \frac{a}{2} a_0^2.$$

$\therefore \boxed{a_0 = \sqrt{\frac{2}{a}}}$. [4.15] $\Rightarrow \psi_{200} = R_{20} Y_0^0$. Table 4.2 $\Rightarrow Y_0^0 = \frac{1}{\sqrt{4\pi}}$. $\therefore \psi_{200} = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{2}{a}}\,\frac{1}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}$

$$\boxed{\psi_{200} = \frac{1}{\sqrt{2\pi a}}\,\frac{1}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}}.$$

(b) $R_{21} = \frac{a_0}{4a^2}\, r e^{-r/2a}$. $\quad 1 = \left(\frac{a_0}{4a^2}\right)^2 a^5 \int_0^\infty z^4 e^{-z}\, dz = \frac{a_0^2 a}{16}\cdot 24 = \frac{3}{2} a\, a_0^2$, so $\boxed{a_0 = \sqrt{\frac{2}{3a}}}$.

$R_{21} = \frac{1}{\sqrt{6a}}\,\frac{1}{2a^2}\, r e^{-r/2a}$. $\quad \psi_{21\pm1} = \frac{1}{\sqrt{6a}}\,\frac{1}{2a^2}\, r e^{-r/2a}\left(\mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\pm i\phi}\right)$

$\boxed{\psi_{21\pm1} = \mp\frac{1}{\sqrt{\pi a}}\,\frac{1}{8a^2}\, r e^{-r/2a} \sin\theta\, e^{\pm i\phi}}$. $\quad \psi_{210} = \frac{1}{\sqrt{6a}}\,\frac{1}{2a^2}\, r e^{-r/2a}\left(\sqrt{\frac{3}{4\pi}}\,\cos\theta\right)$

$$\boxed{\psi_{210} = \frac{1}{\sqrt{2\pi a}}\,\frac{1}{4a^2}\, r e^{-r/2a}\cos\theta}.$$

Problem 4.13 (a) $\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$, so $\langle r^n \rangle = \frac{1}{\pi a^3} \int r^n e^{-2r/a} (r^2 \sin\theta\, dr\, d\theta\, d\phi)$

$$\langle r^n \rangle = \frac{4\pi}{\pi a^3} \int_0^\infty r^{n+2} e^{-2r/a} dr. \quad \langle r \rangle = \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} dr = \frac{4}{a^3} 3! \left(\frac{a}{2}\right)^4 = \boxed{\frac{3}{2}a}.$$

$$\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} dr = \frac{4}{a^3} 4! \left(\frac{a}{2}\right)^5 = \boxed{3a^2}.$$

(b) $\boxed{\langle x \rangle = 0}$; $\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle = \boxed{a^2}$.

(c) $\psi_{211} = R_{21} Y_1^1 = -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta\, e^{i\phi}$. (Problem 4.11(b).)

$$\langle x^2 \rangle = \frac{1}{\pi a} \frac{1}{(8a^2)^2} \int \left(r^2 e^{-r/a} \sin^2\theta\right)\left(r^2 \sin^2\theta \cos^2\phi\right)\left(r^2 \sin\theta\, dr\, d\theta\, d\phi\right)$$

$$= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-r/a} dr \int_0^\pi \sin^5\theta\, d\theta \int_0^{2\pi} \cos^2\phi\, d\phi = \frac{1}{64\pi a^5} \left(6!\, a^7\right)\left(2 \cdot \frac{2\cdot 4}{1\cdot 3\cdot 5}\right)\left(\frac{1}{2}\cdot 2\pi\right)$$

$$= \boxed{12a^2}.$$

Problem 4.14 (a) $P = \int |\Psi|^2 d^3r = \frac{4\pi}{\pi a^3} \int_0^b e^{-2r/a} r^2 dr = \frac{4}{a^3}\left\{-\frac{a}{2} r^2 e^{-2r/a} + \frac{a^3}{4} e^{-2r/a}\left(-\frac{2r}{a} - 1\right)\right\}\Big|_0^b$

$$= -\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)e^{-2r/a}\Big|_0^b = \boxed{1 - \left(1 + \frac{2b}{a} + 2\frac{b^2}{a^2}\right)e^{-2b/a}}.$$

(b) $P = 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right)e^{-\epsilon} \cong 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right)\left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3!}\right) \cong 1 - 1 + \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6} - \epsilon + \epsilon^2 - \frac{\epsilon^3}{2}$

$$-\frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^3 = \epsilon^3\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{6}\epsilon^3 = \frac{1}{6}\left(\frac{2b}{a}\right)^3 = \boxed{\frac{4}{3}\left(\frac{b}{a}\right)^3}.$$

(c) $|\psi(0)|^2 = \frac{1}{\pi a^3} \Rightarrow P \approx \frac{4}{3}\pi b^3 \frac{1}{\pi a^3} = \frac{4}{3}\left(\frac{b}{a}\right)^3$. ✓

(d) $P = \frac{4}{3}\left(\frac{10^{-15}}{.5\times 10^{-10}}\right)^3 = \frac{4}{3}\left(2\times 10^{-5}\right)^3 = \frac{4}{3}\cdot 8\times 10^{-15} = \frac{32}{3}\times 10^{-15} = \boxed{1.07\times 10^{-14}}.$

Problem 4.16 (a) $\boxed{V(r) = -G\frac{Mm}{r}}$. [So $\frac{e^2}{4\pi\epsilon_0} \rightarrow GMm$ to translate hydrogen results.]

(b) [4.72] $\Rightarrow a = \left(\frac{4\pi\epsilon_0}{e^2}\right)\frac{\hbar^2}{m} \rightarrow \boxed{\frac{\hbar^2}{GMm^2}} = \frac{(1.0546\times10^{-34}\ \text{J s})^2}{\left(6.6726\times10^{-11}\frac{\text{m}^3}{\text{kg s}^2}\right)(1.9892\times10^{30}\ \text{kg})(5.98\times10^{24}\ \text{kg})^2}$

$$= \boxed{2.34\times10^{-138}\ \text{m}}$$

(c) [4.70] $\Rightarrow \boxed{E_n = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2}}$. $E_c = \frac{1}{2}mv^2 - G\frac{Mm}{r_0}$. But $G\frac{Mm}{r_0^2} = \frac{mv^2}{r_0} \Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{2r_0}$

So $E_c = -\frac{GMm}{2r_0} = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2} \Rightarrow n^2 = \frac{GMm^2}{\hbar^2}r_0 = \frac{r_0}{a} \Rightarrow \boxed{n = \sqrt{\frac{r_0}{a}}}$.

$r_0 =$ earth-sun distance $= 1.496\times10^{11}\ \text{m} \Rightarrow n = \sqrt{\frac{1.496\times10^{11}}{2.34\times10^{-138}}} = \sqrt{6.39\times10^{148}} = \boxed{2.53\times10^{74}}$.

(d) $\Delta E = -\left[\frac{G^2M^2m^3}{2\hbar^2}\right]\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right]$. $\frac{1}{(n+1)^2} = \frac{1}{n^2(1+\frac{1}{n})^2} \cong \frac{1}{n^2}\left(1-\frac{2}{n}\right)$.

So $\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] \cong \frac{1}{n^2}\left(1-\frac{2}{n}-1\right) = \frac{-2}{n^3}$. $\therefore \Delta E = \frac{G^2M^2m^3}{\hbar^2 n^3}$

$\Delta E = \frac{(6.67\times10^{-11})^2(1.99\times10^{30})^2(5.98\times10^{24})^3}{(1.055\times10^{-34})^2(2.53\times10^{74})^3} = \boxed{2.09\times10^{-41}\ \text{J}}$. $\nu = \frac{E}{h}$; $\lambda = \frac{c}{\nu} = \frac{ch}{E}$.

$\lambda = (3\times10^8)(6.63\times10^{-34})/(2.09\times10^{-41}) = \boxed{9.52\times10^{15}\ \text{m}}$.

PROBLEM 4.17 The potential [4.52] is replaced by $V(r) = -\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$. So $e^2 \to Ze^2$.

$\boxed{E_n(Z) = Z^2 E_n}$ $\boxed{E_1(Z) = Z^2 E_1}$ $\boxed{a(Z) = \frac{1}{Z} a}$ $\boxed{R(Z) = Z^2 R}$. Lyman lines range from $n_i = 2$ to $n_i = \infty$: $\frac{1}{\lambda} = R\left(1 - \frac{1}{4}\right) = \frac{3}{4}R \Rightarrow \lambda = \frac{4}{3R}$; $\frac{1}{\lambda} = R\left(1 - \frac{1}{\infty}\right) = R \Rightarrow \lambda = \frac{1}{R}$.

So for $Z = 2$: $\lambda = \frac{1}{4R} = \frac{1}{4(1.097 \times 10^7)} = \boxed{2.28 \times 10^{-8}\text{ m}}$ to $\lambda = \frac{1}{3R} = \boxed{3.04 \times 10^{-8}\text{ m}}$, $\boxed{\text{ultraviolet}}$;

for $Z = 3$: $\lambda = \frac{1}{9R} = \boxed{1.01 \times 10^{-8}\text{ m}}$ to $\lambda = \frac{4}{27R} = \boxed{1.35 \times 10^{-8}\text{ m}}$ - also $\boxed{\text{ultraviolet}}$.