Time Dependent perturbation theory:

$$\mathcal{H} = H_0 + V(t)$$

define $\quad |\alpha, t_0; t\rangle_I = e^{iH_0 t/\hbar} |\alpha, t_0; t\rangle_S$

define: $\quad A_I \equiv e^{iH_0 t/\hbar} A_S \, e^{-iH_0 t/\hbar}$ (A is arbitrary operator)

so $\quad V_I = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}$

so,
$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle_I = i\hbar \frac{\partial}{\partial t} \left(e^{iH_0 t/\hbar} |\alpha, t_0; t\rangle_S \right)$$
$$= -H_0 e^{iH_0 t/\hbar} |\alpha, t_0; t\rangle_S + e^{iH_0 t/\hbar} (H_0 + V) |\alpha, t_0; t\rangle_S$$
$$= e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} e^{iH_0 t/\hbar} |\alpha\, t_0; t\rangle_S$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle_I = V_I |\alpha, t_0; t\rangle_I \qquad (\star)$$

In the interaction picture we continue using $|n\rangle$, where $|n\rangle$ satisfy

$$H_0 |n\rangle = E_n |n\rangle$$

suppose at $t=0$ $\quad |\alpha\rangle = \sum_n C_n(0) |n\rangle$

now let us find $C_n(t)$ for $t>0$ such that

$$|\alpha, t_0 = 0, t\rangle = \sum_n C_n(t) e^{-iE_n t/\hbar} |n\rangle$$

In the interaction picture $\quad |\alpha, t_0; t\rangle_I = \sum_n C_n(t) |n\rangle \qquad (\star\star)$

Multiplying both sides of $(\star)$ by $\langle n|$ from the left

$$\langle n| e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} |m\rangle = V_{nm}(t) e^{i(E_n - E_m)t/\hbar}$$

and $\quad C_n(t) = \langle n | \alpha\, t_0; t\rangle_I$

$\Rightarrow$ from $(\star\star)$

$$i\hbar \frac{d}{dt} C_n(t) = \sum_m V_{nm} e^{i\omega_{nm} t} C_m(t)$$

where $\quad \omega_{nm} \equiv (E_n - E_m)/\hbar = -\omega_{mn}$

by matrix:

$$i\hbar \begin{pmatrix} \dot c_1 \\ \dot c_2 \\ \dot c_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} e^{i\omega_{12} t} & \cdots & \\ V_{21} e^{i\omega_{21} t} & V_{22} & \cdots & \\ \vdots & \vdots & V_{33} & \\ & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

In perturbation theory, we can get

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} \langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')\,dt'$$

$$c_n^{(2)}(t) = \left(-\frac{i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, e^{i\omega_{nm}t'} V_{nm}(t')\, e^{i\omega_{mi}t''} V_{mi}(t'')$$

$$P(i\to n) = \left|c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots\right|^2$$ when $i \neq n$.

For two level system:

assume $c_a(0) = 1$, $c_b(0) = 0$ for $\langle a|H'(t)|a\rangle = \langle b|H'(t)|b\rangle = 0$

then $c_a^{(0)}(t) = 1$ $\quad c_b^{(0)}(t) = 0$

$$c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^{t} \langle b|H_I'|a\rangle dt'$$

$$= -\frac{i}{\hbar}\int_0^{t} dt'\, e^{i\omega_0 t'} H'_{ba}(t) \qquad \omega_0 = \frac{E_b - E_a}{\hbar}$$

$$\cancel{c_b^{(2)}(t) = -\frac{1}{\hbar^2}\sum_m \int_0^{t} dt' \int_0^{t'} dt''\, e^{i\omega_0 t'} H_{ba}(t')\, e^{i}}$$

$$c_b^{(2)}(t) = -\frac{1}{\hbar^2}\sum_m \int_0^{t} dt' \int_0^{t'} dt''\, e^{i\omega_{nm}t'} V_{nm}(t')\, e^{i\omega_{mi}t''} V_{mi}(t'')$$

~~n=m must be a, b~~

~~but n=b~~ $\quad V_{nm}\cdot V_{mi} = 0.$

So. $c_b^{(2)}(t) = 0.$

$$\Rightarrow \quad |c_b(t)|^2 = \frac{1}{\hbar^2}\left|\int_0^{t} dt'\, e^{i\omega_0 t}\, dt\right|^2 = P_{a\to b}$$

$$P_{aa} = 1 - P_{a\to b}$$

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1. for $a \to b$: (First order):

$$C_b^{(1)}(t) = \langle b | U_I^{(1)}(t) | a \rangle = \frac{-i}{\hbar}\int_0^t dt' \langle b | H_I' | a \rangle.$$

$$= \frac{-i}{\hbar}\int_0^t dt' \langle b | U_0^{+} \hat{H}' U_0 | a \rangle = \frac{-i}{\hbar}\int_0^t dt' \langle b | e^{+iE_b t'/\hbar} \hat{H}' e^{-iE_a t'/\hbar} | a \rangle$$

$$= \frac{-i}{\hbar}\int_0^t dt' \, e^{+iE_b t'/\hbar} \langle b | H' | a \rangle e^{-iE_a t'/\hbar}$$

$$= \frac{-i}{\hbar}\int_0^t dt' \, e^{i(E_b - E_a)t'/\hbar} \langle b | H' | a \rangle$$

$$= \frac{-i}{\hbar}\int_0^t dt' \, e^{i\omega_0 t'} H_{ba}'(t) \qquad \text{where } \omega_0 = \frac{E_b - E_a}{\hbar}.$$

Second Order:

$$C_b^{(2)}(t) = \langle b | U_I^{(2)}(t) | a \rangle = \frac{-1}{\hbar^2}\int_0^t dt'' \int_0^{t''} dt' \langle b | H_I'(t'') H_I'(t') | a \rangle$$

$$H_I'(t'') H_I'(t') = \begin{bmatrix} 0 & H_I'(t'') \\ H_I'^{*}(t'') & 0 \end{bmatrix} \begin{bmatrix} 0 & H_I'(t') \\ H_I'^{*}(t') & 0 \end{bmatrix}$$

$$= \begin{bmatrix} H_I'(t'') H_I'^{*}(t') & 0 \\ 0 & H_I'^{*}(t'') H_I'(t') \end{bmatrix}$$

$$\therefore H_I'(ba) = 0.$$

$$\therefore C_b^{(2)}(t) = 0.$$

$$\therefore C_b(t) = C_b^{(0)}(t) + C_b^{(1)}(t) + C_b^{(2)}(t)$$

$$= 0 + \left[\frac{-i}{\hbar}\int_0^t dt' e^{i\omega_0 t'} H_{ba}'(t)\right] + 0.$$

$$\therefore |C_b(t)|^2 = \frac{1}{\hbar^2}\int_0^t dt' H_{ba}'(t') e^{-i\omega_0 t'} \cdot \left[\int_0^{t'} H_{ba}'(t) e^{+i\omega_0 t''} dt''\right]$$

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∴ for ~~b~~ $\to a$.

$$P_{a\to a} = 1 - P_{b\to a}$$

$$= 1 - \frac{1}{\hbar^2}\int_0^t dt' H'_{\cancel{ab}}(t')\, e^{-i\omega_0 t'} \cdot \left[\int_0^{t'} H'_{ba}(t'')\, e^{+i\omega_0 t''} dt''\right]_{\text{IS}}.$$

Problem 2

$\langle a|H'|a\rangle = \langle b|H'|b\rangle = 0$

$U_I = \mathbb{1} + \lambda U_I^{(1)} + \lambda^2 U_I^{(2)} + \ldots$

$P_{a\to b} = |\langle a|U_I|b\rangle|^2 \qquad P_{a\to a} = |\langle a|U_I|a\rangle|^2$

1^{st} order: $|\langle a|\mathbb{1} + \lambda U_I^{(1)}|b\rangle|^2 = |C_b|^2$

$U_I^{(1)} = -\frac{i}{\hbar}\int_0^t dt' \, H_I'(t)$ (from matching λ after $H_I' \to \lambda H_I'$)

$|C_b|^2 = \langle a|\mathbb{1} + \lambda U_I^{(1)}|b\rangle\langle b|\mathbb{1} + \lambda U_I^{(1)+}|a\rangle$
$= (\langle a|b\rangle + \lambda\langle a|U_I^{(1)}|b\rangle)(\langle b|a\rangle + \lambda\langle b|U_I^{(1)+}|a\rangle)$
$= \lambda^2|\langle a|U_I|b\rangle|^2$

$|C_a|^2 = (\langle a|\mathbb{1} + \lambda U_I^{(1)}|a\rangle)(\langle a|\mathbb{1} + \lambda U_I^{(1)+}|a\rangle)$
$= (\langle a|a\rangle + \lambda\langle a|U_I^{(1)}|a\rangle)(\langle a|a\rangle + \lambda\langle a|U_I^{(1)+}|a\rangle) = 1$

$\therefore |C_b|^2 + |C_a|^2 = 1 + \lambda^2\langle a|U_I|b\rangle|^2$

$\therefore$ unitary is conserved to 1^{st} order ~~but~~ (the order we're working in)

2^{nd} order:

$|C_b|^2 = \langle a|\mathbb{1} + \lambda U_I^{(1)} + \lambda^2 U_I^{(2)}|b\rangle\langle b|\mathbb{1} + \lambda U_I^{(1)+} + \lambda^2 U_I^{(2)+}|a\rangle$
$= (\langle a|b\rangle + \lambda\langle a|U_I^{(1)}|b\rangle + \lambda^2\langle a|U_I^{(2)}|b\rangle)\cdot$
$(\langle b|a\rangle + \lambda\langle b|U_I^{(1)+}|a\rangle + \lambda^2\langle b|U_I^{(2)+}|a\rangle)$
$= (\lambda\langle a|U_I^{(1)}|b\rangle + \lambda^2\langle a|U_I^{(2)}|b\rangle)(\lambda\langle b|U_I^{(1)+}|a\rangle + \lambda^2\langle b|U_I^{(2)+}|a\rangle)$
$= \lambda^2|\langle a|U_I^{(1)}|b\rangle|^2 + \lambda^3\langle a|U_I^{(1)}|b\rangle\langle b|\cancel{U_I^{(2)+}}|a\rangle_0 \ldots$ p302

$|C_a|^2 = (\langle a|a\rangle + \lambda\langle a|U_I^{(1)}|a\rangle + \lambda^2\langle a|U_I^{(2)}|a\rangle)\cdot$
$(\langle a|a\rangle + \lambda\langle a|U_I^{(1)+}|a\rangle + \lambda^2\langle a|U_I^{(2)+}|a\rangle)$

$|C_a|^2 = (1 + \lambda^2\langle a|U_I^{(2)}|a\rangle)(1 + \lambda^2\langle a|U_I^{(2)+}|a\rangle)$

$|C_a|^2 = 1 + \lambda^2\langle a|U_I^{(2)+}|a\rangle + \lambda^2\langle a|U_I^{(2)}|a\rangle + \lambda^4 \ldots$

$|C_b|^2 + |C_a|^2 = \lambda^2|\langle a|U_I^{(1)}|b\rangle|^2 + 1 + \lambda^2\langle a|U_I^{(2)}|a\rangle + \lambda^2\langle a|U_I^{(2)+}|a\rangle$

$\frac{d}{dt}(|C_b|^2 + |C_a|^2) = \lambda^2\left(\frac{\partial}{\partial t}|\langle a|U_I^{(1)}|b\rangle|^2 + \frac{\partial}{\partial t}|\langle a|U_I^{(2)}|a\rangle| + \frac{\partial}{\partial t}|\langle a|U_I^{(2)+}|a\rangle\right)$

$= \lambda^2\left[\frac{\partial}{\partial t}\left|-\frac{i}{\hbar}\int_0^t dt' e^{i(E_a-E_b)t'/\hbar} H'_{ab}(t')\right|^2 + \frac{\partial}{\partial t}\left(-\frac{i}{\hbar}\right)^2\int_0^t dt'' e^{i(E_a-E_b)t''/\hbar} H'_{ab}(t'') \left(\int_0^{t''} dt' e^{i(E_b-E_a)t'/\hbar} H'_{ba}(t')\right)\right.$

$\left. + \frac{\partial}{\partial t}\left(-\frac{i}{\hbar}\right)\int_0^t dt'' e^{-i(E_a-E_b)t''/\hbar} H'_{ba}(t'') \frac{i}{\hbar}\int_0^{t''} dt' e^{-i(E_b-E_a)t'/\hbar} H'_{ab}(t')\right)$

$= \lambda^2\left(\frac{\partial}{\partial t}|C_b^{(1)}|^2 + \frac{\partial}{\partial t}C_a^{(2)} + \frac{\partial}{\partial t}C_a^{*(2)}\right)$

$= \lambda^2\left(C_b\frac{\partial}{\partial t}C_b^* + C_b^*\frac{\partial}{\partial t}C_b + \frac{\partial}{\partial t}\left(-\frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}C_b\,dt'\right) + \frac{\partial}{\partial t}\left(\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}C_b\,dt'\right)\right.$

$= \lambda^2\left(C_b\cdot\frac{i}{\hbar}H'_{ab}(t)e^{-i\omega_0 t} - \frac{i}{\hbar}H'_{ab}(t)e^{-i\omega_0 t'}C_b + \frac{i}{\hbar}H'_{ba}(t)e^{i\omega_0 t}C_b^*\right.$

$\left. + C_b^*\cdot -\frac{i}{\hbar}H'_{ba}(t)e^{i\omega_0 t}\right) = 0$ ✓

$\therefore |C_b|^2 + |C_a|^2 = 1$ to second order

b/c @ $t=0$, $|C_b|^2 = 0$ & $|C_a|^2 = 1$ $\therefore |C_a|^2 + |C_b|^2 = 1$ @ $t=0$ & all time

Problem 9.6 For H' independent of t, [9.17] $\Rightarrow c_b^{(2)}(t) = c_b^{(1)}(t) = -\frac{i}{\hbar}H'_{ba}\int_0^t e^{i\omega_0 t'}dt' \Rightarrow$

$$c_b^{(2)}(t) = -\frac{i}{\hbar}H'_{ba}\,\frac{e^{i\omega_0 t'}}{i\omega_0}\Big|_0^t = \boxed{-\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t}-1\right)}.$$ Meanwhile [9.18] $\Rightarrow$

$$c_a^{(2)}(t) = 1 - \frac{1}{\hbar^2}|H'_{ab}|^2\int_0^t e^{-i\omega_0 t'}\Big\{\underbrace{\int_0^{t'} e^{i\omega_0 t''}dt''}_{\frac{e^{i\omega_0 t''}}{i\omega_0}\Big|_0^{t'} = \frac{1}{i\omega_0}\left(e^{i\omega_0 t'}-1\right)}\Big\}dt' = 1 - \frac{1}{\hbar^2}|H'_{ab}|^2\frac{1}{i\omega_0}\int_0^t\left(1-e^{-i\omega_0 t'}\right)dt'$$

$$c_a^{(2)}(t) = 1 + \frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left(t' + \frac{e^{-i\omega_0 t'}}{i\omega_0}\right)\Big|_0^t = \boxed{1+\frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left[t+\frac{1}{i\omega_0}\left(e^{-i\omega_0 t}-1\right)\right]}.$$

For comparison with the exact answers (Problem 9.2), note first that $c_b(t)$ is already first order (because of the H'_{ba} in front), whereas ω differs from ω_0 only in second order, so it suffices to replace $\omega \to \omega_0$ in the exact formula to get the second-order result:

$$c_b(t) \cong \frac{2H'_{ba}}{i\hbar\omega_0}e^{i\omega_0 t/2}\sin(\omega_0 t/2) = \frac{2H'_{ba}}{i\hbar\omega_0}e^{i\omega_0 t/2}\frac{1}{2i}\left(e^{i\omega_0 t/2}-e^{-i\omega_0 t/2}\right) = -\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t}-1\right),$$ in agreement with the result above. Checking c_a is more difficult. Note that

$$\omega = \omega_0\sqrt{1+\frac{4|H'_{ab}|^2}{\omega_0^2\hbar^2}} \cong \omega_0\left(1+2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2}\right) = \omega_0 + \frac{2|H'_{ab}|^2}{\omega_0\hbar^2};\quad \frac{\omega_0}{\omega} \cong 1-2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2};$$

Taylor expansion:
$$\begin{cases} \cos(x+\epsilon) = \cos x - \epsilon\sin x \Rightarrow \cos(\omega t/2) = \cos\left(\frac{\omega_0 t}{2} + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\right) \cong \cos(\omega_0 t/2) - \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\sin(\omega_0 t/2) \\ \sin(x+\epsilon) = \sin x + \epsilon\cos x \Rightarrow \sin(\omega t/2) = \sin\left(\frac{\omega_0 t}{2} + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\right) \cong \sin(\omega_0 t/2) + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\cos(\omega_0 t/2) \end{cases}$$

Putting this into the exact expression (Problem 9.2), we expand to second order in H:

$$c_a(t) \cong e^{-i\omega_0 t/2}\left\{\cos\left(\frac{\omega_0 t}{2}\right) - \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\sin(\omega_0 t/2) + i\left(1-2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2}\right)\left[\sin\left(\frac{\omega_0 t}{2}\right) + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\cos(\omega_0 t/2)\right]\right\}$$

$$= e^{-i\omega_0 t/2}\left\{\left[\cos\left(\frac{\omega_0 t}{2}\right) + i\sin\left(\frac{\omega_0 t}{2}\right)\right] - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[t\left(\sin\left(\frac{\omega_0 t}{2}\right) - i\cos\left(\frac{\omega_0 t}{2}\right)\right) + \frac{2i}{\omega_0}\sin(\omega_0 t/2)\right]\right\}$$

$$= e^{-i\omega_0 t/2}\left\{e^{i\omega_0 t/2} - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[-it\,e^{i\omega_0 t/2} + \frac{2i}{\omega_0}\frac{1}{2i}\left(e^{i\omega_0 t/2}-e^{-i\omega_0 t/2}\right)\right]\right\}$$

$$= 1 - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[-it + \frac{1}{\omega_0}\left(1-e^{-i\omega_0 t}\right)\right] = 1 + \frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left[t + \frac{1}{i\omega_0}\left(e^{-i\omega_0 t}-1\right)\right],$$ as above. ✓

Problem 9.7 (a) $\dot c_a = -\frac{i}{2\hbar}V_{ab}e^{i\omega t}e^{-i\omega_0 t}c_b$; $\dot c_b = -\frac{i}{2\hbar}V_{ba}e^{-i\omega t}e^{i\omega_0 t}c_a$. Differentiate the latter, and substitute in the former:

$$\ddot{c}_b = -i\frac{V_{ba}}{2\hbar}\left[i(\omega_0-\omega)e^{i(\omega_0-\omega)t}c_a + e^{i(\omega_0-\omega)t}\dot{c}_a\right]$$

$$= i(\omega_0-\omega)\left[-i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}c_a\right] - i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}\left[-\frac{i}{2\hbar}V_{ab}e^{-i(\omega_0-\omega)t}c_b\right] = i(\omega_0-\omega)\dot{c}_b - \frac{|V_{ab}|^2}{(2\hbar)^2}c_b.$$

$\frac{d^2c_b}{dt^2} + i(\omega-\omega_0)\frac{dc_b}{dt} + \frac{|V_{ab}|^2}{4\hbar^2}c_b = 0$. Solution is of the form $c_b = e^{\lambda t}$: $\lambda^2 + i(\omega-\omega_0)\lambda + \frac{|V_{ab}|^2}{4\hbar^2} = 0 \Rightarrow$

$\lambda = \frac{1}{2}\left[-i(\omega-\omega_0) \pm \sqrt{-(\omega-\omega_0)^2 - |V_{ab}|^2/\hbar^2}\right] = i\left[-\frac{(\omega-\omega_0)}{2} \pm \omega_r\right]$, with ω_r defined in [9.30].

General solution: $c_b(t) = Ae^{i\left[-\frac{(\omega-\omega_0)}{2}+\omega_r\right]t} + Be^{i\left[\frac{(\omega-\omega_0)}{2}-\omega_r\right]t} = e^{-i(\omega-\omega_0)t/2}\left[Ae^{i\omega_r t} + Be^{-i\omega_r t}\right]$,

or, more conveniently: $c_b(t) = e^{-i(\omega-\omega_0)t/2}\left[C\cos(\omega_r t) + D\sin(\omega_r t)\right]$. But $c_b(0) = 0$, so $C = 0$:

$c_b(t) = De^{i(\omega_0-\omega)t/2}\sin(\omega_r t)$. $\therefore \dot{c}_b = D\left\{i\left(\frac{\omega_0-\omega}{2}\right)e^{i(\omega_0-\omega)t/2}\sin(\omega_r t) + \omega_r e^{i(\omega_0-\omega)t/2}\cos(\omega_r t)\right\}$; so

$c_a(t) = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t}\dot{c}_b = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t/2}D\left[i\left(\frac{\omega_0-\omega}{2}\right)\sin(\omega_r t) + \omega_r\cos(\omega_r t)\right]$. But $c_a(0) = 1$, so

$1 = i\frac{2\hbar}{V_{ba}}D\omega_r$, or $D = -\frac{iV_{ba}}{2\hbar\omega_r}$. $\therefore$ $\boxed{c_b(t) = -\frac{i}{2\hbar\omega_r}V_{ba}e^{i(\omega_0-\omega)t/2}\sin(\omega_r t)}$, and

$\boxed{c_a(t) = e^{i(\omega-\omega_0)t/2}\left[\cos(\omega_r t) + i\left(\frac{\omega_0-\omega}{2\omega_r}\right)\sin(\omega_r t)\right]}$.

(b) $P_{a\to b}(t) = |c_b(t)|^2 = \boxed{\left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t)}$. The largest this gets (when $\sin^2 = 1$) is $\frac{|V_{ab}|^2/\hbar^2}{4\omega_r^2}$,

and the denominator, $4\omega_r^2 = (\omega-\omega_0)^2 + |V_{ab}|^2/\hbar^2$, exceeds the numerator, so $P \leq 1$ (and 1 only if $\omega = \omega_0$).

$|c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \left(\frac{\omega_0-\omega}{2\omega_r}\right)^2\sin^2(\omega_r t) + \left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t) = \cos^2(\omega_r t) + \frac{(\omega-\omega_0)^2 + (|V_{ab}|/\hbar)^2}{4\omega_r^2}\sin^2(\omega_r t)$

$= \cos^2(\omega_r t) + \sin^2(\omega_r t) = 1$ ✓.

(c) If $\boxed{|V_{ab}|^2 \ll \hbar^2(\omega-\omega_0)^2}$, then $\omega_r \cong \frac{1}{2}|\omega-\omega_0|$, and $P_{a\to b} \cong \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega-\omega_0)^2}$, confirming [9.28].

(d) $\omega_r t = \pi \to \boxed{t = \pi/\omega_r}$.