PROBLEM 6.13 [6.52] $\Rightarrow$ $E_r^1 = -\frac{1}{2mc^2}\left[E^2 - 2E\langle V\rangle + \langle V^2\rangle\right]$. Here $E = (n+\frac{1}{2})\hbar\omega$, $V = \frac{1}{2}m\omega^2x^2$.

$\therefore E_r^1 = -\frac{1}{2mc^2}\left[(n+\frac{1}{2})^2\hbar^2\omega^2 - 2(n+\frac{1}{2})\hbar\omega\frac{1}{2}m\omega^2\langle x^2\rangle + \frac{1}{4}m^2\omega^4\langle x^4\rangle\right]$. But Problem 2.37 $\Rightarrow \langle x^2\rangle = (n+\frac{1}{2})\frac{\hbar}{m\omega}$,

so $E_r^1 = -\frac{1}{2mc^2}\left[(n+\frac{1}{2})^2\hbar^2\omega^2 - (n+\frac{1}{2})^2\hbar^2\omega^2 + \frac{1}{4}m^2\omega^4\langle x^4\rangle\right] = -\frac{m\omega^4}{8c^2}\langle x^4\rangle$. Again from Problem 2.37:

$x^4 = \frac{1}{4m^2\omega^4}(a_+^2 - a_+a_- - a_-a_+ + a_-^2)(a_+^2 - a_+a_- - a_-a_+ + a_-^2)$, so

$\langle x^4\rangle = \frac{1}{4m^2\omega^4}\langle n|(a_+^2a_-^2 + a_+a_-a_+a_- + a_+a_-a_-a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ + a_-^2a_+^2)|n\rangle$ (note that only terms with equal numbers of raising and lowering operators will survive).

$$\langle x^4\rangle = \frac{1}{4m^2\omega^4}\langle n|\Big\{a_+^2\left(\sqrt{n(n-1)}\,\hbar\omega|n-2\rangle\right) + a_+a_-(n\hbar\omega|n\rangle) + a_+a_-((n+1)\hbar\omega|n\rangle) + a_-a_+(n\hbar\omega|n\rangle)$$
$$+ a_-a_+((n+1)\hbar\omega|n\rangle) + a_-^2\left(\sqrt{(n+1)(n+2)}\,\hbar\omega|n+2\rangle\right)\Big\}$$
$$= \frac{\hbar}{4m^2\omega^3}\langle n|\Big\{\sqrt{n(n-1)}\left(\sqrt{n(n-1)}\,|n\rangle\right)\hbar\omega + n(n\hbar\omega|n\rangle) + (n+1)(n\hbar\omega|n\rangle) + n((n+1)\hbar\omega|n\rangle)$$
$$+ (n+1)((n+1)\hbar\omega|n\rangle) + \sqrt{(n+1)(n+2)}\left(\sqrt{(n+1)(n+2)}\,\hbar\omega|n\rangle\right)\Big\}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left\{n(n-1) + n^2 + (n+1)n + n(n+1) + (n+1)^2 + (n+1)(n+2)\right\}$$
$$= \left(\frac{\hbar}{2m\omega}\right)^2\left(n^2 - n + n^2 + n^2 + n + n^2 + n + n^2 + 2n + 1 + n^2 + 3n + 2\right) = \left(\frac{\hbar}{2m\omega}\right)^2(6n^2 + 6n + 3)$$

$$\therefore E_r^1 = -\frac{m\omega^4}{8c^2}\cdot\frac{\hbar^2}{4m^2\omega^2}\cdot 3(2n^2+2n+1) = \boxed{-\frac{3}{32}\left(\frac{\hbar^2\omega^2}{mc^2}\right)(2n^2+2n+1)}.$$

PROBLEM 6.14 (a) $[\vec{L}\cdot\vec{S}, L_x] = [L_xS_x + L_yS_y + L_zS_z, L_x] = S_x[L_x, L_x] + S_y[L_y, L_x] + S_z[L_z, L_x]$
$= S_x(0) + S_y(-i\hbar L_z) + S_z(i\hbar L_y) = i\hbar(L_yS_z - L_zS_y) = i\hbar(\vec{L}\times\vec{S})_x$.

Same goes for the other two components, so $\boxed{[\vec{L}\cdot\vec{S}, \vec{L}] = i\hbar(\vec{L}\times\vec{S})}$. (b) $[\vec{L}\cdot\vec{S}, \vec{S}]$ is identical, only with $\vec{L}\leftrightarrow\vec{S}$: $\boxed{[\vec{L}\cdot\vec{S}, \vec{S}] = i\hbar(\vec{S}\times\vec{L})}$. (c) $[\vec{L}\cdot\vec{S}, \vec{J}] = [\vec{L}\cdot\vec{S}, \vec{L}] + [\vec{L}\cdot\vec{S}, \vec{S}] = i\hbar(\vec{L}\times\vec{S} + \vec{S}\times\vec{L}) = \boxed{0}$.

(d) L^2 commutes with all components of $\vec{L}$ (and $\vec{S}$), so $\boxed{[\vec{L}\cdot\vec{S}, L^2] = 0}$. (e) Likewise $\boxed{[\vec{L}\cdot\vec{S}, S^2] = 0}$.

(f) $[\vec{L}\cdot\vec{S}, J^2] = [\vec{L}\cdot\vec{S}, L^2] + [\vec{L}\cdot\vec{S}, S^2] + 2[\vec{L}\cdot\vec{S}, \vec{L}\cdot\vec{S}] = 0 + 0 + 0 \Rightarrow \boxed{[\vec{L}\cdot\vec{S}, J^2] = 0}$.

<u>Problem 6.17</u> $\sqrt{(j+1/2)^2-\alpha^2} = (j+1/2)\sqrt{1-(\alpha/(j+1/2))^2} \cong (j+1/2)\left[1-\frac{1}{2}\left(\frac{\alpha}{j+1/2}\right)^2\right] = (j+\frac{1}{2}) - \frac{\alpha^2}{2(j+1/2)}$.

$$\therefore \frac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}} \cong \frac{\alpha}{n-(j+1/2)+(j+\frac{1}{2})-\frac{\alpha^2}{2(j+1/2)}} = \frac{\alpha}{n-\frac{\alpha^2}{2(j+1/2)}} = \frac{\alpha}{n\left[1-\frac{\alpha^2}{2n(j+1/2)}\right]}$$

$$\cong \frac{\alpha}{n}\left[1+\frac{\alpha^2}{2n(j+1/2)}\right].$$

$$\left[1+\left(\frac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}}\right)^2\right]^{-1/2} \cong \left[1+\frac{\alpha^2}{n^2}\left(1+\frac{\alpha^2}{n(j+1/2)}\right)\right]^{-1/2} \cong 1-\frac{1}{2}\frac{\alpha^2}{n^2}\left(1+\frac{\alpha^2}{n(j+1/2)}\right)+\frac{3}{8}\frac{\alpha^4}{n^4}$$

$$= 1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4}{2n^4}\left(\frac{-n}{j+1/2}+\frac{3}{4}\right).$$

$$\therefore E_{nj} \cong mc^2\left\{1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4}{2n^4}\left(\frac{-n}{j+1/2}+\frac{3}{4}\right)-1\right\} = \underbrace{-\frac{\alpha^2 mc^2}{2n^2}}_{E_n \text{ (Problem 6.10)}}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right]$$

$$= -\frac{13.6\,\text{eV}}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right],$$ confirming [6.66].

CHAPTER 9

PROBLEM 9.1 $\psi_{nlm} = R_{nl} Y_l^m$. From Tables 4.2 and 4.5:

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \; ; \; \psi_{200} = \frac{1}{\sqrt{8\pi a^3}}\left(1-\frac{r}{2a}\right)e^{-r/2a} \; ; \; \psi_{210} = \frac{1}{\sqrt{32\pi a^3}}\frac{r}{a} e^{-r/2a}\cos\theta ; \; \psi_{21\pm1} = \mp\frac{1}{\sqrt{64\pi a^3}}\frac{r}{a}e^{-r/2a}\sin\theta e^{\pm i\phi}.$$

But $r\cos\theta = z$ and $r\sin\theta e^{\pm i\phi} = r\sin\theta(\cos\phi \pm i\sin\phi) = r\sin\theta\cos\phi \pm i r\sin\theta\sin\phi = x \pm iy$. So $|\psi|^2$ is an even function of z in all cases, and hence $\int z|\psi|^2 dx\,dy\,dz = 0$. $\therefore$ $\boxed{H'_{ii} = 0}$. Moreover, ψ_{100} is even in z, and so are ψ_{200}, ψ_{211}, and ψ_{21-1}, so $\boxed{H'_{ij} = 0}$ for all except

$$H'_{100,210} = -eE\frac{1}{\sqrt{\pi a^3}}\frac{1}{\sqrt{32\pi a^3}}\frac{1}{a}\int e^{-r/a}e^{-r/2a}z^2 d^3r = -\frac{eE}{4\sqrt{2}\pi a^4}\int e^{-3r/2a} r^2\cos^2\theta\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= -\frac{eE}{4\sqrt{2}\pi a^4}\int_0^\infty r^4 e^{-3r/2a}dr\int_0^\pi \cos^2\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi = -\frac{eE}{4\sqrt{2}\pi a^4}\cdot 4!\left(\frac{2a}{3}\right)^5\cdot\frac{2}{3}\cdot 2\pi = \boxed{-\left(\frac{2^8}{\sqrt{2}\,3^5}\right)eEa},$$

or $\boxed{-0.7449\, eEa}$.

PROBLEM 9.2 $\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \; ; \; \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a$.

$\therefore \ddot{c}_b = -\frac{i}{\hbar}H'_{ba}\left[i\omega_0 e^{i\omega_0 t}c_a + e^{i\omega_0 t}\dot{c}_a\right] = i\omega_0\left[-\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a\right] - \frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}\left[-\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b\right]$, or

$\ddot{c}_b = i\omega_0\dot{c}_b - \frac{1}{\hbar^2}|H'_{ab}|^2 c_b$. Let $\alpha^2 \equiv \frac{1}{\hbar^2}|H'_{ab}|^2$. Then $\ddot{c}_b - i\omega_0\dot{c}_b + \alpha^2 c_b = 0$. This is a linear differential equation with constant coefficients, so it can be solved by a function of the form $c_b = e^{\lambda t}$:

$\lambda^2 - i\omega_0\lambda + \alpha^2 = 0 \Rightarrow \lambda = \frac{1}{2}\left[i\omega_0 \pm \sqrt{-\omega_0^2 - 4\alpha^2}\right] = \frac{i}{2}\left[\omega_0 \pm \omega\right]$, where $\omega \equiv \sqrt{\omega_0^2 + 4\alpha^2}$. The general solution is therefore $c_b(t) = Ae^{\frac{i}{2}(\omega_0+\omega)} + Be^{\frac{i}{2}(\omega_0-\omega)} = e^{i\omega_0 t/2}\left(Ae^{i\omega t/2} + Be^{-i\omega t/2}\right)$, or

$c_b(t) = e^{i\omega_0 t/2}\left[C\cos(\omega t/2) + D\sin(\omega t/2)\right]$. But $c_b(0) = 0$, so $C = 0$, and hence $c_b(t) = De^{i\omega_0 t/2}\sin(\omega t/2)$.

$\therefore \dot{c}_b = D\left\{\frac{i\omega_0}{2}e^{i\omega_0 t/2}\sin(\omega t/2) + \frac{\omega}{2}e^{i\omega_0 t/2}\cos(\omega t/2)\right\} = \frac{\omega}{2}De^{i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right] = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a$.

$\therefore c_a = \frac{i\hbar}{H'_{ba}}\frac{\omega}{2}e^{-i\omega_0 t/2}D\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right]$. But $c_a(0) = 1$, so $\frac{i\hbar}{H'_{ba}}\frac{\omega}{2}D = 1$.

$$\therefore \boxed{\begin{aligned} c_a(t) &= e^{-i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right] \\ c_b(t) &= \frac{2H'_{ba}}{i\hbar\omega}e^{i\omega_0 t/2}\sin(\omega t/2)\end{aligned}}, \text{ where } \boxed{\omega \equiv \sqrt{\omega_0^2 + 4|H'_{ab}|^2/\hbar^2}}.$$

$$|c_a|^2 + |c_b|^2 = \cos^2(\omega t/2) + \frac{\omega_0^2}{\omega^2}\sin^2(\omega t/2) + \frac{4|H'_{ab}|^2}{\hbar^2\omega^2}\sin^2(\omega t/2) = \cos^2(\omega t/2) + \frac{1}{\omega^2}\left[\omega_0^2 + 4|H'_{ab}|^2/\hbar^2\right]\sin^2(\omega t/2)$$

$$= \cos^2(\omega t/2) + \sin^2(\omega t/2) = 1 \checkmark$$

PROBLEM 9.3 [9.13] $\Rightarrow \frac{dc_a}{dt} = -\frac{i\alpha}{\hbar}\delta(t-t_0)e^{-i\omega_0 t}c_b \Rightarrow c_a(t) = c_a(-\infty) - \frac{i\alpha}{\hbar}\int_{-\infty}^{t}\delta(t'-t_0)e^{-i\omega_0 t'}c_b(t')\,dt' \Rightarrow$

$c_a(t) = 1 - \frac{i\alpha}{\hbar}e^{-i\omega_0 t_0}c_b(t_0)\theta(t-t_0)$. Likewise $\frac{dc_b}{dt} = -\frac{i\alpha^*}{\hbar}\delta(t-t_0)e^{i\omega_0 t}c_a \Rightarrow$

$c_b(t) = c_b(-\infty) - \frac{i\alpha^*}{\hbar}\int_{-\infty}^{t}\delta(t'-t_0)e^{i\omega_0 t'}c_a(t')\,dt' = -\frac{i\alpha^*}{\hbar}e^{i\omega_0 t_0}c_a(t_0)\,\theta(t-t_0)$. This is slippery, ~~because~~ c_a and c_b are discontinuous at t_0, but we obtain consistent results by interpreting $\theta(0)$ as $\frac{1}{2}$; then

$c_a(t_0) = 1 - \frac{i\alpha}{2\hbar}e^{-i\omega_0 t_0}c_b(t_0)$; $c_b(t_0) = -\frac{i\alpha^*}{2\hbar}e^{i\omega_0 t_0}c_a(t_0)$, so $c_a(t_0) = 1 - \frac{|\alpha|^2}{4\hbar^2}c_a(t_0)$, or

$c_a(t_0) = \frac{1}{1+|\alpha|^2/4\hbar^2}$; $c_b(t_0) = -\frac{i\alpha^*}{2\hbar}\frac{e^{i\omega_0 t_0}}{1+|\alpha|^2/4\hbar^2}$. Putting these back into the formulas for c_a and c_b

$$\boxed{c_a(t) = 1 - \left(\frac{|\alpha|^2/2\hbar^2}{1+|\alpha|^2/4\hbar^2}\right)\theta(t-t_0)}\;;\;\boxed{c_b(t) = -\frac{i\alpha^*}{\hbar}e^{i\omega_0 t_0}\frac{1}{1+|\alpha|^2/4\hbar^2}\theta(t-t_0)}.$$

For $t<t_0$, $c_a(t)=1$, $c_b(t)=0$, so obviously $|c_a|^2+|c_b|^2=1$.

For $t>t_0$, $c_a(t) = 1 - \frac{|\alpha|^2/2\hbar^2}{1+|\alpha|^2/4\hbar^2} = \frac{1-|\alpha|^2/4\hbar^2}{1+|\alpha|^2/4\hbar^2}$, $c_b(t) = -\frac{i\alpha^*}{\hbar}\frac{e^{i\omega_0 t_0}}{(1+|\alpha|^2/4\hbar^2)}$, so

$$|c_a|^2+|c_b|^2 = \frac{(1-|\alpha|^2/2\hbar^2+|\alpha|^4/16\hbar^4)+|\alpha|^2/\hbar^2}{(1+|\alpha|^2/4\hbar^2)^2} = \frac{(1+|\alpha|^2/4\hbar^2)^2}{(1+|\alpha|^2/4\hbar^2)^2} = 1 \checkmark$$

The probability of a transition is $|c_b|^2 = \boxed{\frac{|\alpha|^2/\hbar^2}{(1+|\alpha|^2/4\hbar^2)^2}}$.