Now often we can get in the ball park with a one parameter family and <u>very</u> close with a few

~~∂~~

$$|\Psi(\alpha_1, \alpha_2 \cdots \alpha_n)\rangle$$

$$\frac{\partial \langle\Psi(\alpha_1, \alpha_2 \cdots \alpha_n)|\hat{H}|\Psi(\alpha_1 \cdots \alpha_n)\rangle}{\partial \alpha_n} = 0$$

the reason is simple

once $|\Psi_{approx}\rangle$ is close to $|\Psi_{true}\rangle$

$$|\Psi_{approx}\rangle = |\Psi_{true}\rangle + \lambda|\delta\Psi\rangle + \mathcal{O}(\lambda^2)$$

where $\langle\delta\Psi|\Psi_{true}\rangle = 0$ (normalization

then

$$\langle\Psi_{approx}|\hat{H}|\Psi_{approx}\rangle =$$

$$\underbrace{\langle\Psi_{true}|\hat{H}|\Psi_{true}\rangle}_{\epsilon_1} + \lambda\langle\delta\Psi|\hat{H}|\Psi_{true}\rangle + \lambda\langle\Psi_{true}|\hat{H}|\delta\Psi\rangle + \mathcal{O}(\lambda^2)$$

but $\langle \delta\Psi | \hat{H} | \Psi_{true} \rangle = \langle \delta\Psi | E | \Psi_{true} \rangle = E \langle \delta\Psi | \Psi_{true} \rangle$
$= 0$

$$\langle \Psi_{approx} | \hat{H} | \Psi_{approx} \rangle = \langle \Psi_{true} | \hat{H} | \Psi_{true} \rangle + O(\lambda^2)$$

$$= E_{true} + O(\lambda^2)$$

No error of order λ!!

One nice trick; it often a pain to stick to normalized trial functions; you can use unnormalized ones provided you minimize function

$$\frac{\langle \Psi(\alpha) | \hat{H} | \Psi(\alpha) \rangle}{\langle \Psi(\alpha) | \Psi(\alpha) \rangle}$$

example H.O. $\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$

trial function

$\langle x|\psi(\alpha)\rangle = \psi(x) = \begin{cases} (a^2 - x^2)^2 & |x| \leq a \\ 0 & |x| \geq a \end{cases}$

O.k. form as ψ, ψ' are cont. at a

$$E(\alpha) = \frac{\langle\psi(\alpha)|\hat{H}|\psi(\alpha)\rangle}{\langle\psi(\alpha)|\psi(\alpha)\rangle}$$

$$= \frac{\int_{-a}^{a} dx \left(\frac{-\hbar^2}{2m} \psi^* \psi'' + \frac{1}{2} m\omega^2 x^2 \psi^* \psi\right)}{\int_{-a}^{a} dx\, \psi^* \psi}$$

$$= \underbrace{\frac{3\hbar}{2m\alpha^2}}_{\text{kinetic}} + \underbrace{\frac{1}{22} a^2 m \omega}_{\text{potential}}$$

Note trade off ~~and~~ between T and U

large α minimizes T
small α minimizes U
$\Big\}$ comprimise

min

$$0 = \frac{\partial E}{\partial \alpha} = \frac{-3\hbar^2}{a^3} + \frac{1}{11} a m w^2$$

or $$a_{min} = \frac{33^{\frac{1}{4}} \sqrt{\hbar}}{\sqrt{m w}}$$

plug in

$$E(a_{min}) = \sqrt{\frac{3}{11}}\, \hbar w$$

now $\sqrt{\frac{3}{11}} = .522$

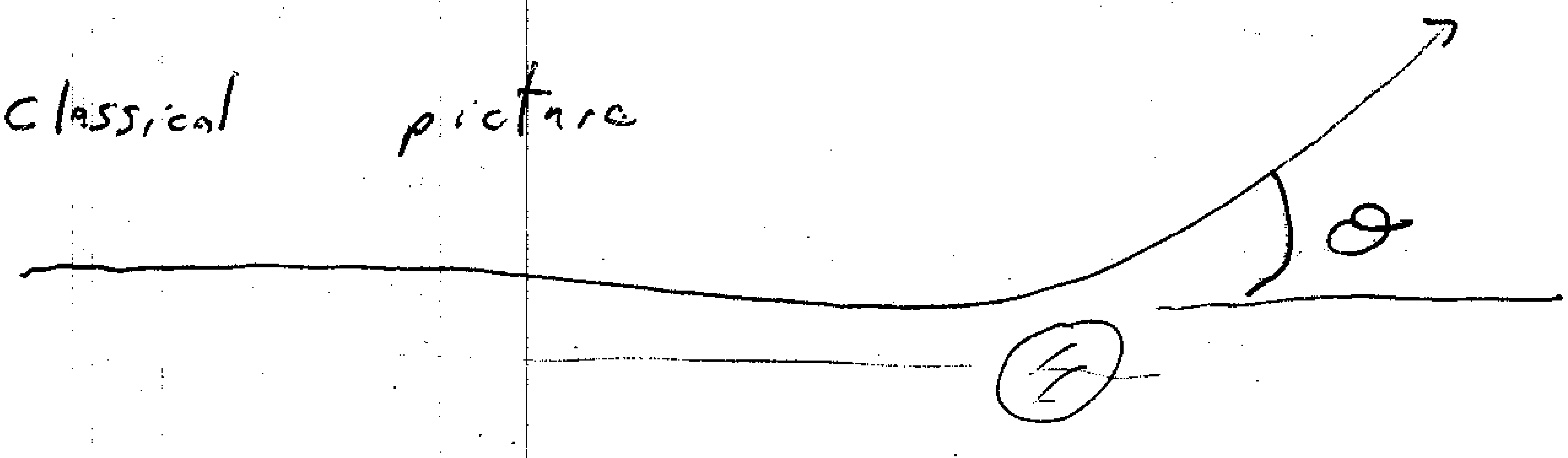
$$E_g^{approx} \approx .522\, \hbar w \quad > \quad E_g^{exact} = .5\, \hbar w$$

Scattering theory —

Simplest case: a spinless particle scattering off a fixed potential $V(\vec{x})$ with $V(\infty)=0$

clearly if a particle comes in with $E>0$ it cannot get stuck so it goes back to ∞

classical picture



θ is a scattering angle — the angle between incident and scattered particle

Classically this depends on impact parameter (how close to middle line I sent in) but in Q.M. this ain't defined

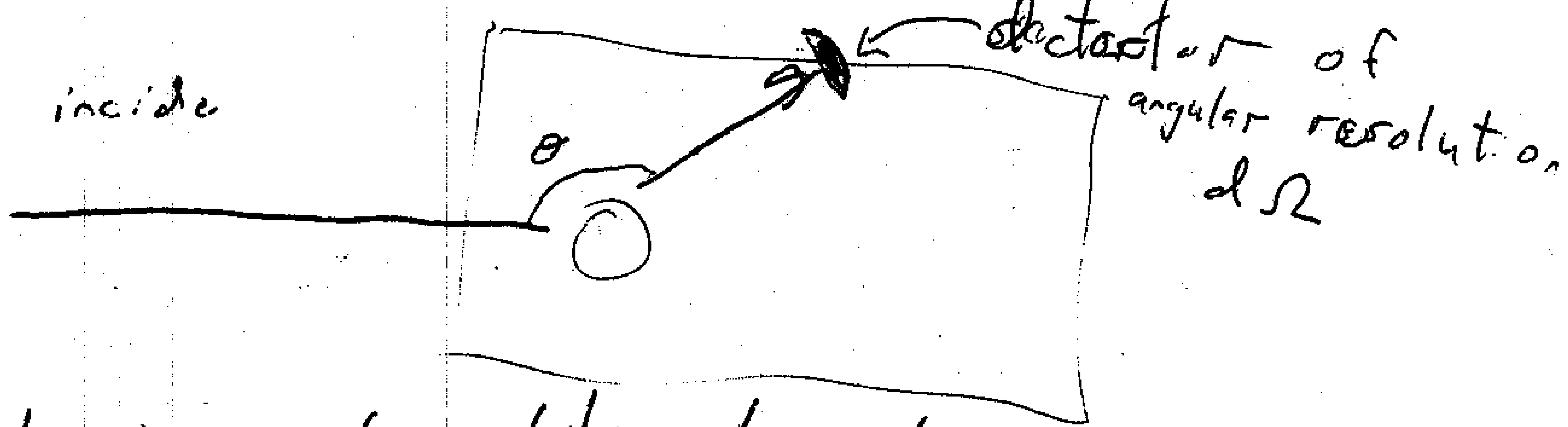
How should an experimentor proceed?

Shoot a flux of particles at target

Flux = # of particles / area / time

do this was known incident momentum and energy

particles will be scattered and come out various angles (generally take two angles θ, ϕ to describe Ω (solid angle is short hand for θ, ϕ



ϕ is angle relative to plane

put a detector at fixed Ω and count # of particles coming into $d\Omega$ per unit times

- Rate is proportional to incident flux
  why

- Rate is proportional to $d\Omega$ (why)

define

$$\frac{d\sigma}{d\Omega} = \frac{\text{counting rate of particles into dector / per unit solid angle}}{\text{Flux}}$$

$\sim \dfrac{\text{number of particles / time}}{\text{number of particles/time/area}} \sim$. area

differential cross-section

$\dfrac{d\sigma}{d\Omega}$ is a function of E (or p) and Ω

natural experimental quantity

total cross-section

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{\text{\# of collisons per time}}{\text{flux}}$$

"shadow size" but is a function of E

actually there is a subtlety here

how can I distinguish a "miss" from forward scattering

if the potential is ∞ ranged (eg. colomb) I can't and $\sigma = \infty$ if I take the $\int d\Omega \frac{d\sigma}{d\Omega}$. I always scatter (a bit of Coulomb)

Classically if $V(r) = 0$ for all $r > a$ (some #) then I can distinguish hits from misses in Q.M. same is true if $V(r) = 0$ beyond some range

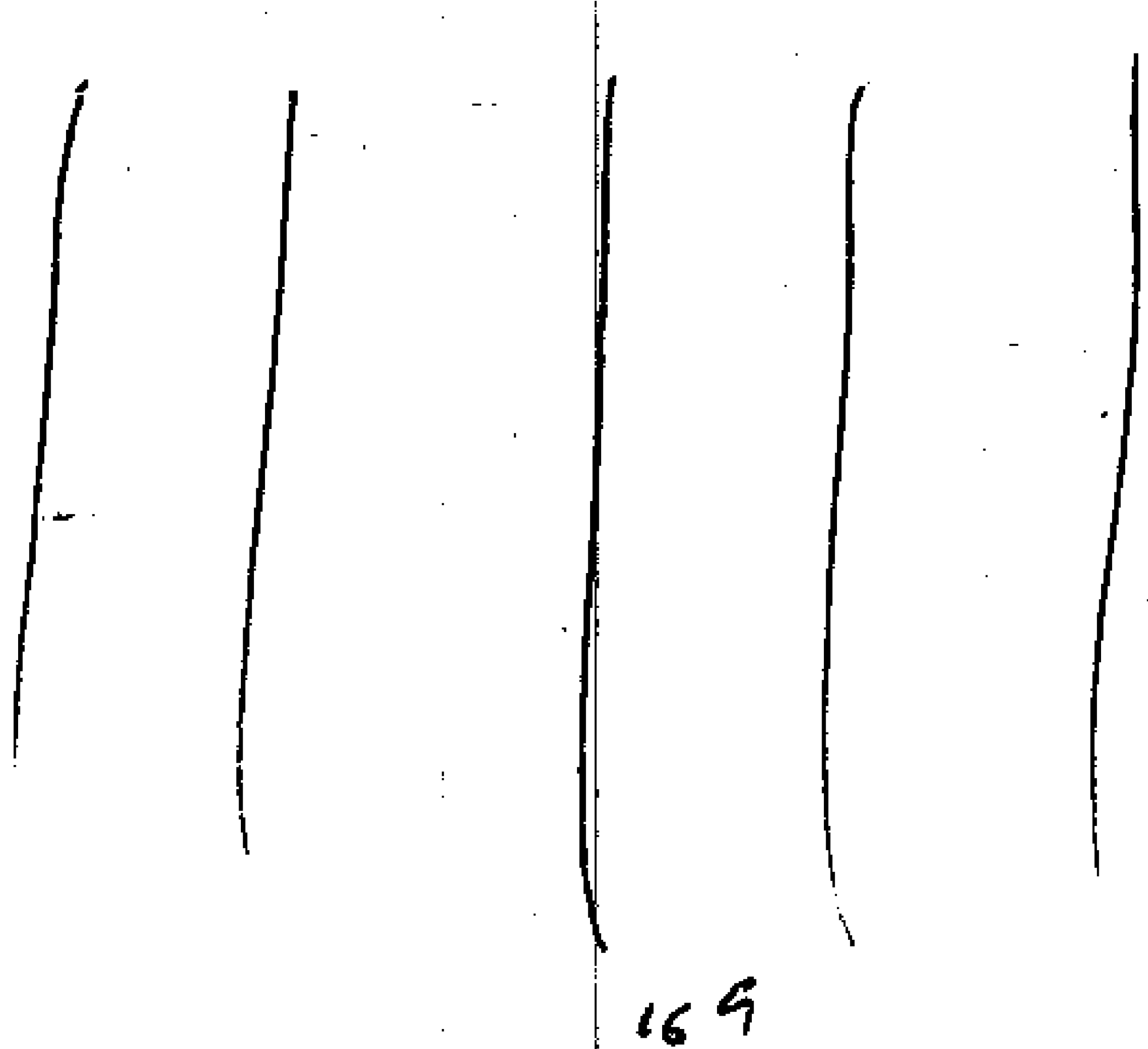
$$\int \frac{d\sigma}{d\Omega}\, d\Omega \qquad \text{is finite}$$

actually I just require $V(r)$ to fall off "fast enough" as $r \to \infty$ (formally
any power law faster than $1/r$

if it is fast enough beyond some range
it is small enough to neglect

first let me describe a flux
with no scattering: namely a plane
wave

$$\psi(x) = A\, e^{i(kz - \omega t)} \qquad \hbar\omega = \frac{(\hbar k)^2}{2m}$$



$E = \hbar\omega$
$P = \hbar k$

generically ~~scatter~~ incident in any direction; conventionally usually take as z

what is flux

Claim: Prob density $\psi^* \psi = |A|^2$
~~each particle~~ particle has a velocity = $\frac{\hbar k}{m}$

$\frac{\hbar k}{m}$

flux = Prob density × v = $\frac{\hbar k}{m} |A|^2$
why?

Now what does wavefunction look like if there is scattering

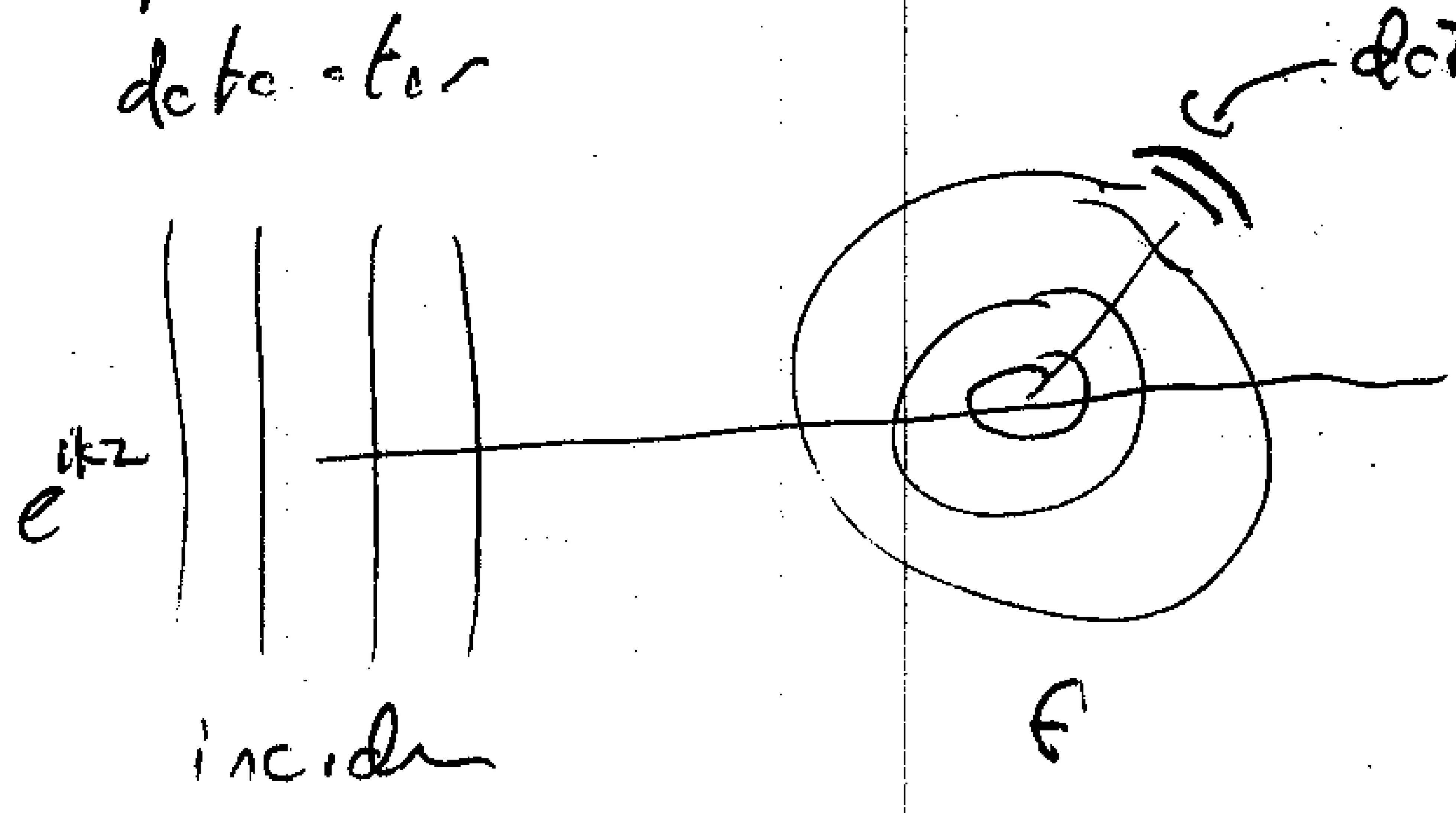
suppose potential is localized at $r=0$ and $V(r) = 0$ for $r > a$

then f

~~eq~~ solves free Schrödinger equation ~~with~~ with bc. that it is a plane wave incident from $-z$ and an outgoing spherical wave (angular dependent from origin)
(solution with ingoing solves Schröd. eq. but of physical interest)

Now $f(\Omega) \equiv$ scattering amplitude

Now what is the probability that the outgoing ~~part~~ will flow into my detector



rate of flow outward ~~...~~ Area per time

$$\rho_{scattered} v_{out} = |\psi^*_{scat} \psi_{scat}| \frac{\hbar k}{m} = \frac{|A|^2 |f(\Omega)|^2}{r^2} \frac{\hbar k}{m}$$

Now $r^2 d\Omega = dA$

so # of particles per unit time per unit solid angle

detected/time/$d\Omega$ $= \rho_{scat} v_{out} * \frac{dA}{d\Omega} = |A|^2 |f(\Omega)|^2 \frac{\hbar k}{m}$

Now $\frac{d\sigma}{d\Omega} = \frac{\#\text{ detected}/\text{time}/d\Omega}{\text{flux}} = \frac{|A|^2 |f(\Omega)|^2 \hbar k/m}{|A|^2 \hbar k/m}$

$= |f(\Omega)|^2$

to find differential cross section find $f(\Omega)$

Now here is the problem

$\Psi(\vec{r},t) = \psi(\vec{r})\, e^{-iEt/\hbar}$

Find $\psi(\vec{r})$ with b.c. that as $r \to \infty$ looks like planewave plus outgoing

$\psi(\vec{r}) \to A\left[e^{ikz} + f(\Omega)\frac{e^{ikr}}{r}\right]$

trouble is how do you solve Schrödinger equation with ~~flux~~ b.c.

- start studying simplest case central potential

$$V(\vec{r}) = V(r)$$

- partial wave expansion (useful for small energies

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(r)\Psi = E\Psi$$

$$\Psi = \sum_{lm} R(r)\, Y_{lm}(\theta,\phi)\, c_{lm}$$

$$u = r R(r)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u(r) = E$$

as $r \to \infty$ $V \to 0$ $\cancel{\frac{l(l+1)}{r^2} \to 0}$

$$\cancel{\frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u = -k^2 u}$$

$$\cancel{u = C e^{ikr} + D e^{-ikr}}$$

So

$$\frac{\partial^2 u}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} u = k^2 u \quad \text{with } k^2 = \frac{2mE}{\hbar^2}$$

Now solutions are the spherical bessel and Neuman functions

(Not J_ℓ, N_ℓ)

$u = r\, j_\ell(kr)$ regular at 0 $\qquad r\, n_\ell(kr)$ divergent at zero

$j_0(x) = \frac{\sin(x)}{x}$

$n_0(x) = -\cos(x)$

$n_1(x) = \frac{\cos x}{x^2} - \frac{\sin x}{x}$

$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos(x)}{x}$

$n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin$

$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3}{x^2}\cos(x)$

But these don't look like incoming our outgoing waves (when you add $e^{i\omega t}$)

take linear combos

$h_\ell^1(x) = j_\ell(x) + i n_\ell(x) \qquad h_\ell^2 = h_\ell^{1*}$

$h_0^1(x) = -i \frac{e^{ix}}{x}$

$h_1^1 = \left(-\frac{i}{x^2} - \frac{1}{x}\right) e^{ix}$

$h_2^1 = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{ix}$

Note as $r \to \infty$

$$h'_\ell = (-i)^{l+1} \frac{e^{ikr}}{r}$$

outgoing wave

so in region where $V=0$

$$\psi(r, \Omega) = A\left[e^{ikz} + \sum_{\ell m} c_{\ell m} h_\ell^{(1)}(kr) Y_\ell^m(\Omega)\right]$$

Claim if $V(r)$ is spherically sym then total problem is Azimuthal sym about z and all ϕ are created equal: $c_{\ell m} = 0$ unless $m=0$

$$Y_\ell^0(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

For $V=0$

$$\psi(r,\theta) = A\left[e^{ikz} + \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}}\, c_\ell\, h_\ell^{(1)}(kr) P_\ell(\cos\theta)\right]$$

as $r \to \infty$ $\quad h_\ell^{(1)}(kr) = (-i)^{\ell+1} \frac{e^{ikr}}{r}$

$$\psi(r,\theta) = A\left[e^{ikz} + \sum_{\ell=0}^{\infty} (-i)^{\ell+1} \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)\, c_\ell \frac{e^{ikr}}{r}\right]$$

but as $r \to \infty$

$$\psi = A\left[e^{ikz} + f(\theta) \frac{e^{ikr}}{r}\right]$$

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (-i)^{l+1} \sqrt{\frac{2l+1}{4\pi}}\ c_l\ P_l(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2} \sum_{l,l'} c_l c_{l'}^* (-1)^{l+l'} \frac{\sqrt{(2l+1)(2l'+1)}}{4\pi} P_l(\cos\theta) P_{l'}(\cos\theta)$$

and

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int_0^{\pi} \sin\theta\, d\theta \quad \frac{d\sigma}{d\Omega}$$

use fact that $\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \frac{2}{2l+1} \delta_{l,l'}$

so

$$\sigma = \frac{1}{k^2} \sum_l \sum_{l'} c_l c_{l'}^* (-1)^{l+l'} \frac{\sqrt{(2l+1)(2l'+1)}}{2l+1} \delta_{l,l'}$$

$$= \frac{1}{k^2} \sum_l |c_l|^2$$

Problem is now how to compute these c_l

Note we need to solve Sch. eq subject to b.c. of incident plane wave + outgoing spherical wave.

we cannot fully exploit spherical sym since incident plane wave is not written in terms of l

but there is a marvelous identity due to Rayleigh

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos(\theta))$$

so we can now relate incident wave in terms of partial wave

note that $\quad j_l(kr) = \frac{1}{2}\left(h_l^{(1)} + h_l^{(2)}\right)$

outgoing spherical $\qquad$ incoming spherical claim unitarity says flux in equals flux out

so for large r

$$\Psi \approx \frac{A}{k} \sum_{\ell=0}^{\infty} i^{\ell} (2\ell+1) P_\ell(\cos\theta) \left\{ \frac{(+i)^{\ell+1}}{2} \frac{e^{-ikr}}{r} + \underbrace{e^{2i\delta_\ell}}_{\text{convention}} \frac{(-i)^{\ell+1}}{2} \frac{e^{ikr}}{r} \right\}$$

effect of potential in each partial wave described entirely by phase shift δ_ℓ

comparing

$$\frac{e^{2i\delta_\ell}}{2} = \left(\frac{1}{2} + \frac{c_\ell (-i)^{\ell}}{\sqrt{2\ell+1}\,\sqrt{4\pi}} \right)$$

or

$$c_\ell = i^{\ell} \sqrt{2\ell+1}\,\sqrt{4\pi}\,\tfrac{1}{2}\left(e^{2i\delta_\ell} - 1\right)$$

$$= i^{\ell} \sqrt{2\ell+1}\,\sqrt{4\pi}\, e^{i\delta_\ell}\, \tfrac{1}{2}\left(e^{i\delta_\ell} - e^{-i\delta_\ell}\right)$$

$$= i^{\ell+1} \sqrt{2\ell+1}\,\sqrt{4\pi}\, e^{i\delta_\ell} \sin(\delta_\ell)$$

129 ~~magnitude of~~ c_ℓ fixed by δ_ℓ ; max of $|c_\ell| = \sqrt{4\pi(2\ell+1)}$

Plug back into

$$f(\theta) = \frac{1}{k} \sum_{l=0} (-i)^{l+1} \sqrt{\frac{2l+1}{4\pi}}\, c_l P_l(\cos\theta)$$

σ

$$\sigma = \frac{1}{k^2} \sum_l |c_l|^2$$

$$\boxed{f(\theta) = \frac{1}{k} \sum_{l=0} (2l+1)\, e^{i\delta_l} \sin(\delta_l)\, P_l(\cos\theta)}$$

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l}$$

scattering observables fixed by phase shifts

still how do I compute phase shifts !!?

First neat factoid

look at

$$f(\theta=0) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\,e^{i\delta_l}\sin(\delta_l)\,\underset{\substack{\parallel \\ 1}}{P_l(1)}$$

$$\text{Im}\,f(\theta=0) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

but

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

$$\therefore \quad \boxed{\sigma = \frac{4\pi}{k}\,\text{Im}\,f(\theta=0)}$$

Optical theorem

trick in general.

solve Schrödinger equation partial wave by partial wave

$$\frac{-\hbar^2}{2m}\frac{\partial^2 u_\ell}{\partial r^2} + \left(V + \frac{\hbar^2 \ell(\ell+1)}{2m}\right) u_\ell = \frac{\hbar^2 k^2}{2m} u_\ell$$

Now to do this we need correct b.c.

it is a 2^{nd} order equation so there are two b.c. but one is just an overall normalization (if u_ℓ is a solution Au_ℓ is too).

thus we need to impose one non trivial b.c. this at $r=0$, in general there are two classes of solutions (regular at $r=0$ and divergent at $r=0$) we pick the regular one.

this gives us

$u_\ell(r)$ for all r up to an overall scale which doesn't matter (why)

We have solution everywhere.

but let us recall that general solution for r big enough so $V(r)\approx 0$ or is negligible

$$\Psi(r) = A\sum_{\ell}\left(i^{\ell}(2\ell+1)j_\ell(kr) + \sqrt{\tfrac{2\ell+1}{4\pi}}\,C_\ell\, h_\ell^{(1)}(kr)\right) P_\ell(\cos\theta)$$

thus for r big enough

$$u_\ell(r) = A\left(i^{\ell}(2\ell+1)j_\ell(kr) + \sqrt{\tfrac{2\ell+1}{4\pi}}\,C_\ell\, h_\ell^{(1)}(kr)\right)$$

Now we can match the true wave function onto this form to find the C_ℓ

method look at

$$\alpha_\ell = \left.\frac{u_\ell'}{u_\ell}\right|_R$$

this removes dependence on A

R is big enough so $V(R)$ is negligible

- evaluate for true wave function

- asymptotically this can also be evaluated

$$\frac{u_l'}{u_l} = \frac{i^l(2l+1)\,k\,j_l'(kR) + \sqrt{\frac{2l+1}{4\pi}}\,k\,h_l^{(1)\prime}(kR)\,c_l}{i^l(2l+1)\,j_l(kR) + \sqrt{\frac{2l+1}{4\pi}}\,h_l^{(1)}(kR)\,c_l}$$

equating gives

$$\alpha_l = \frac{k\left(j_l'(kR) + \frac{c_l(-i)^l}{\sqrt{(2l+1)4\pi}}\,h_l^{(1)\prime}(kR)\right)}{\left(j_l(kR) + \frac{c_l(-i)^l}{\sqrt{(2l+1)4\pi}}\,h_l^{(1)}(kR)\right)}$$

or

$$\cancel{c_l = \alpha_l(j_l(kR)) + k\,j_l'(kR)}$$

$$\frac{(-i)^l}{\sqrt{4\pi(2l+1)}}\,c_l = \frac{\left(-\alpha_l\,j_l(kR) + k\,j_l'(kR)\right)}{\left(\alpha_l\,h_l^{(1)}(kR) + k\,h_l^{(1)\prime}(kR)\right)}$$

or

$$c_l = i^l\sqrt{4\pi(2l+1)}\;\frac{\left(-\alpha_l\,j_l(kR) + k\,j_l'(kR)\right)}{\left(\alpha_l\,h_l^{(1)}(kR) + k\,h_l^{(1)\prime}(kR)\right)}$$

solving Schrödinger equation fixes c_l

recall

$$c_\ell = i^{\ell+1}\sqrt{(2\ell+1)4\pi}\; e^{i\delta_\ell}\,(\sin\delta_\ell)$$

so fixing c_ℓ gives phase shift

$\therefore$ solving Schrödinger eq. with b.c. of regular solution at $r=0$ fixes c_ℓ and thus describes f, $\frac{d\sigma}{d\Omega}$ & σ (the physical observables)

eg. hard sphere

$$V(r) = \begin{cases} +\infty & r \le a \\ 0 & r > a \end{cases}$$

really limit of finite potential
effect is to make $\psi(r,\theta) = 0$ for $r \le a$

at boundary $r = a$

$$\psi(a,\theta) = 0 = \sum_{\ell=0}^{\infty}\left(i^\ell(2\ell+1)j_\ell(ka) + \sqrt{\tfrac{2\ell+1}{4\pi}}\, c_\ell\, h_\ell^{(1)}(ka)\right) P_\ell(\cos\theta) = 0$$

clearly term in ( ) $= 0$ for each ℓ (why)
integrate both sides by $P_{\ell'}(\cos\theta)$ and $\int$

thus we don't have to use α_ℓ trick immediately get

$$C_\ell = (-i)^\ell \sqrt{4\pi(2\ell+1)} \; \frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}$$

for $ka \ll 1$ only $\ell = 0$ important

$$C_0 = \sqrt{4\pi} \; \frac{j_0(ka)}{h_\ell^{(1)}(ka)} = \sqrt{4\pi} \; \frac{\sin(ka)/ka}{-i\, e^{ika}/ka}$$

$$= i\sqrt{4\pi}\; e^{-ika} \sin(ka)$$

NOW $\quad \sigma = \frac{1}{k^2} \sum_{\ell=0}^{\infty} |C_\ell|^2 \simeq \frac{1}{k^2} |C_0|^2 = \frac{4\pi}{k^2} \sin^2(ka)$

only valid for $ka \ll 1$ so $\sin^2(ka) \simeq (ka)^2$

$$\sigma \approx \frac{4\pi}{k^2} k^2 a^2 = 4\pi a^2$$

becomes exact as $k \to 0$

what about large k

Formal Solution to scattering theory & Born approx

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\Psi = E\Psi$$

or

$$(\nabla^2 + k^2)\,\Psi(\vec{r}) = \frac{2m}{\hbar^2} V(r)\,\Psi(\vec{r})$$

Use Green's function method

suppose we can find $G(\vec{r})$

$$(\nabla^2 + k^2)\,G(\vec{r}) = \delta^3(r) \qquad \text{subject to b.c.}$$

• then

$$(\nabla^2 + k^2)\,\Psi(\vec{r}) = \frac{2m}{\hbar^2} V(r)\,\Psi(\vec{r})$$

is solved by kz

$$\Psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int d^3r'\, G(\vec{r}-\vec{r}\,')\,V(r')\,\Psi(r')$$

proof act on both sides by

$\nabla^2 + k^2$ get

$$(\nabla^2 + k^2)\,\psi(\vec r) = \cancel{(\nabla^2+k^2)\,e^{ikz}} + \frac{2m}{\hbar^2}\int d^3r'\,(\nabla^2+k^2)\,G(\vec r - r')\,V(r')\,\psi(r')$$

$$= \frac{2m}{\hbar^2}\int d^3r'\,\delta(\vec r - \vec r')\,V(r')\,\psi(r')$$

$$= \frac{2m}{\hbar^2}\,V(\vec r')\,\psi(r)$$

Q.E.D

Claim I know G

$$G(r) = \frac{-1}{4\pi}\,\frac{e^{ikr}}{r}$$

proof in the book is much too complicated there is a simple derivation (which I will skip

so

$$\psi(r) = e^{i\vec k \vec r} + \frac{2m}{\hbar^2}\int d^3r'\,\frac{e^{-k|r-r'|}}{|r-r'|}\,\psi(r')\,V(r')$$

unfortunate this is hard to solve (its an integral equation

Suppose though effect of scattering of total wave function is small

V is small (compared to what

E is high

$V \to \lambda V$

$$\Psi = \Phi e^{i\vec{k}\cdot\vec{r}} + \lambda \Psi_1 + \lambda^2 \Psi_2 + \cdots$$

plug back into $\Psi = e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int d^3r' \, G(\vec{r}-\vec{r}')V\Psi$

$$\Psi_0 + \lambda\Psi_1 + \lambda^2\Psi_2 + \cdots =$$

$$e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int d^3r' \, G(\vec{r}-\vec{r}')\lambda V(e^{i\vec{k}\cdot\vec{r}} + \lambda\Psi_1 + \lambda^2\Psi_2 + \cdots)$$

equate powers

λ^0 $\quad \Psi_0 = e^{i\vec{k}\cdot\vec{r}}$

$\lambda^{(1)}$ $\quad \Psi_1 = \frac{2m}{\hbar^2}\int d^3r' \, G(\vec{r}-\vec{r}')V(r')e^{i\vec{k}\vec{r}}$

$\Psi^{(2)}$ $\quad \Psi_2 = \frac{2m}{\hbar^2}\int d^3r' \, G(\vec{r}-\vec{r}')V(r')\Psi_1(\vec{r}') =$

$$\left(\frac{2m}{\hbar^2}\right)^2 \int d^3r' \int d^3r'' \, G(r''-r')V(r'') \, G(r'-r)V(r')e^{ik\cdot r}$$

Born Series

look at first Born approx

$$\psi_1(r) = \left(\frac{2m}{\hbar^2}\right) \int d^3r' \frac{-e^{ik|\vec{r}-\vec{r}'|}}{4\pi\,|\vec{r}-\vec{r}'|} V(r')\, e^{ik\cdot r'}$$

how does this look as $r \to \infty$

$$|r - \vec{r}'| \to r - \frac{\vec{r}\cdot\vec{r}'}{r} = r - \hat{n}\cdot\vec{r}' + O\left(\left(\frac{r'}{r}\right)^2\right)$$

in denominator $\hat{n}\cdot r'$ is negligible in exponent it matters

large r

$$\psi_1 = -\left(\frac{2m}{\hbar^2}\right) \frac{e^{ikr}}{4\pi r} \int d^3r'\, V(r')\, e^{i\vec{k}\cdot\vec{r}' - i\hat{n}\cdot\vec{r}' k}$$

but we can define $\vec{k}' = \hat{n} k$ (wave vector towards detector)

$$\psi_1 = \frac{-m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3r'\, V(r')\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}$$

Note in this approx

$$\psi = e^{i\vec{k}\cdot\vec{r}} - \left(\frac{m}{2\pi\hbar^2}\right)\left(\int d^3r'\, V(r')\, e^{i(\vec{k}-\vec{k}')\cdot r'}\right)\frac{e^{ikr}}{r}$$

but in general

$$\psi = e^{i\vec{k}\cdot\vec{r}} + f(r)\,\frac{e^{ikr}}{r}$$

so

$$f(\Omega) = \frac{-m}{2\pi\hbar^2}\int d^3r'\, V(r')\, e^{i(\vec{k}-\vec{k}')\cdot r}$$

up to constants

$f(\Omega)$ is FT of potential evaluated at $\vec{q} = \vec{k}-\vec{k}'$ $\hbar\vec{q}$ is the momentum transfer