construct explicit wave functions

for $n=1$ must be $l=0$ as $n = K_{max}+l+1$
Ground state must be $k_{max}=0$ $\therefore$ $k=0$
Note groundstate has $l=0$ compare with Bohr!
only one term in

$$v(\rho) = \sum_{k=0}^{0} a_k \rho^k = a_0$$

so $$\underset{n\ l}{u_{1,0}} = a_0 \overset{l+1}{\rho^1} e^{-\rho} \qquad \rho = \kappa r = r/a$$ ← Bohr

$$u_{10} = \frac{a_0 r}{a} e^{-r/a}$$

$$R_{1,0} = \frac{u_{1,0}}{r} = \frac{a_0}{a} e^{-r/a}$$

a_0 fixed Normalization

$$\int \overbrace{r^2 dr\, d\Omega}^{d^3r} |\Psi|^2 = 1 \qquad \Psi_{1,0,0} = R_{1,0} Y_0^0$$

$$\int r^2 dr\, R_{1,0}(r) \underbrace{\int d\Omega\, |Y_{00}|^2}_{1}$$

so $\int_0^\infty |R_{10}|^2 r^2 dr = 1$

$$1 = \int_0^\infty dr\; r^2 \frac{|a_0|^2}{a^2} e^{-2r/a} = |a_0|^2 \frac{a}{4}$$

$$a_0 = \frac{2}{\sqrt{a}} \qquad Y_{00}^0 = \frac{1}{\sqrt{4\pi}}$$

or

$$\boxed{\psi_{1,0,0} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}}$$

what about $n = 2$

two cases $l = 1$ and $l = 0$

$l = 0$ case: $k_{max} = 1$ $\qquad K_{max} = n - l - 1$

$$a_{k-1} = \frac{k(k+2l+1)}{2(k+l) - 2n} a_k$$

$$a_0 = \frac{1(2)\, a_1}{2(1) - 2(2)} = -a_1$$

$$R = \frac{\rho^{l+1} e^{-\rho} \sum_k^1 a_k \rho^k}{r} \qquad \rho = \frac{r}{a_n}$$

$$R_{2,0} = \frac{r\, e^{-r/2a}}{(2a)\, r} \times a_0 - \frac{a_0 r}{2a}$$

$$= \frac{a_0}{2a} e^{-r/2a} \left(1 - \frac{r}{2a}\right)$$

a_0 is fixed by normalization

what about $l=1$; $R_{2,1}$

$k_{max} = n - l - 1 = 0$ one term

$$R_{2,0} = \frac{\rho^{l+1} e^{-\rho}\, a_0}{r} \qquad \rho = \frac{r}{na}$$

$$= \frac{a_0\, r\, e^{-r/2a}}{4a^2}$$

again a_0 fixed by norm

You can find any you want

general form $R_{n,l} = \text{const}\; e^{-r/na} \times$ Polynomial in $\frac{r}{na}$ of degree $n-1$

Polynomials are associate Laguerre Polynomials (see book)

Spin

One of the over simplifications of the hydrogen atom was ~~ignori~~ the neglect of the spin of the electron (or proton)

What is spin?

Intrinsic angular momentum of a particle (not associated with orbital motion)

History Uhlenbeck & Goldsmit (Krönig) introduced on phenomenological grounds (need extra degree of freedom to describe spectra)

analogy — motion of a rigid object in classical mechanics (say earth orbiting sun)

angular momentum has two components

$$\vec{J} = \vec{L} + \vec{S}$$

total angular momentum

angular momentum of center of mass

angular momentum of object spinning about center of mass electron acts similarly — there is an intrinsic spin about the "center of electron"

(subtle point the electron is "point-like" but still has ~~other~~ spin angular momentum)

Formally spin behaves like orbital angular momentum; commutators define algebra

$$\boxed{[\hat{S}_i, \hat{S}_j] = i\,\varepsilon_{ijk}\,\hbar\,\hat{S}_k}$$

from this we deduce

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \qquad \hat{S}_\pm = \hat{S}_x \pm i\,\hat{S}_y$$

$$\boxed{\begin{aligned} \hat{S}^2 |s, m\rangle &= \hbar^2 s(s+1)\,|s\ m\rangle \\ \hat{S}_z |s, m\rangle &= \hbar\, m\,|s, m\rangle \\ \hat{S}_\pm |s, m\rangle &= \hbar\sqrt{s(s+1) - m(m\pm 1)}\ |s\ (m\pm 1)\rangle \end{aligned}}$$

what values can s take
from consistency of commutators
and positivity of the norm for $\hat{L}$
we deduced $\hat{L}$ must be integer or $\frac{1}{2}$ integer
but we ruled out $\frac{1}{2}$ integer as being inconsistent
with orbital wave function ϕ. but that

argument does not apply to intrinsic motion

$\therefore \quad s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ all possible

$m = -s, -s+1, -s+2, \cdots, s-1, s$

~~most important cases~~

each species of particle has a fixed value for s

all electrons are spin $\frac{1}{2}$

(as are positrons, μ's, τ's, quarks, protons, neutrons ...)

all photons are spin 1

(as are $W^{\pm}$, Z, Gluons, ρ mesons)

all pions are spin 0

(as are α particles, Higgs ...)

different from classical mech — a ball doesn't cease to be a ball when it spins faster

Let's study spin $\frac{1}{2}$ (most important case: all fundamental matter particles are spin $\frac{1}{2}$)

two spin states $|\frac{1}{2}\ \frac{1}{2}\rangle$ and $|\frac{1}{2}\ -\frac{1}{2}\rangle$
(for simplicity often indicated $|\uparrow\rangle$ and $|\downarrow\rangle$

generic state

$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ $\qquad |a|^2 + |b|^2 = 1$

$\langle\uparrow|\psi\rangle = a$
$\langle\downarrow|\psi\rangle = b$

Now I can represent $|\psi\rangle$ as a two element column vector

$$X = \begin{pmatrix} \langle\uparrow|\psi\rangle \\ \langle\downarrow|\psi\rangle \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$ has all the info

about $|\psi\rangle$

$$X^\dagger = (a^* \ \ b^*) = (\langle\psi|\uparrow\rangle \ \ \langle\psi|\downarrow\rangle)$$

$$X^\dagger X = 1$$

Now what are $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$ action on $|\psi\rangle$

define by action on $|\uparrow\rangle$ and $|\downarrow\rangle$

eg $\quad S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \qquad$ or $\qquad S_z \chi = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\chi$

$\quad S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

$$S_x|\uparrow\rangle = \tfrac{1}{2}(S_+ + S_-)|\uparrow\rangle = \tfrac{1}{2}\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\downarrow\rangle = \tfrac{\hbar}{2}|\downarrow\rangle$$

$$S_x|\downarrow\rangle = \tfrac{1}{2}(S_+ + S_-)|\downarrow\rangle = \tfrac{1}{2}\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)}\;|\uparrow\rangle = \tfrac{\hbar}{2}|\uparrow\rangle$$

$$S_x \chi = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\chi$$

$$S_y|\uparrow\rangle = \frac{1}{2i}(S_+ - S_-)|\uparrow\rangle = \frac{\hbar}{2}\, i\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\downarrow\rangle = \frac{\hbar}{2}\, i\,|\downarrow\rangle$$

$$S_y|\uparrow\rangle = -\frac{\hbar}{2}\, i\,|\downarrow\rangle$$

or

$$S_y \chi = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\chi$$

or acting on χ

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

claim easy to see that

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

in the sense of matrices

which implies

$$[\hat{S}_i, \hat{S}_j] = i\hbar\,\varepsilon_{ijk}\,\sigma_k \qquad \text{(homework)}$$

Meaning of $\hat{S}_x, \hat{S}_y\ \hat{S}_z$

- eigvalues of $S_z = \pm\frac{1}{2}\hbar$ trivial
- eigenvalue of S_x, S_y also $\pm\frac{1}{2}\hbar$

easy to see this λ is an eigenvalue of S_x if

$$\text{if } \det\left[-\lambda \mathbb{1} + \underbrace{S_x}_{\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}}\right] = 0$$

$$\text{or } 0 = \begin{vmatrix} -\lambda & \hbar/2 \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \qquad \lambda = \pm\frac{\hbar}{2} \quad \text{Q.E.D}$$

similar for S_y

Eigen functions for S_x, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

check $\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ $\qquad \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\-\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$

call $\begin{pmatrix}1/\sqrt{2}\\\frac{1}{\sqrt{2}}\end{pmatrix} \equiv |\uparrow\rangle_x \qquad \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix} \equiv |\downarrow\rangle_x$

if you measure a spin $\frac{1}{2}$ particle along <u>any</u> axis you will get either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

eg. suppose

$$\chi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{5}}|\uparrow\rangle + \frac{2}{\sqrt{5}}|\downarrow\rangle$$

Prob. of getting $|\uparrow\rangle = |\langle\uparrow|\psi\rangle|^2 = \frac{1}{5}$
in z direction. Prob of $|\downarrow\rangle = |\langle\downarrow|\psi\rangle|^2 = \frac{4}{5}$

what about in x

$$|\psi\rangle = a\,|\uparrow\rangle_x + b\,|\downarrow\rangle_x$$

$$a = {}_x\langle\uparrow|\psi\rangle = \left(\tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = \frac{3}{\sqrt{10}}$$

$$b = {}_x\langle\downarrow|\psi\rangle = \left(\tfrac{1}{\sqrt{2}}\ \tfrac{-1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = \frac{-1}{\sqrt{10}}$$

Prob of $|\uparrow\rangle$ measured in x is $a^2 = 9/10$
Prob of $|\downarrow\rangle$ $b^2 = 1/10$

General comment

c.m. motion and spin described by

$|\psi\rangle = \sum_m \int d\vec{x}\ \psi_m(x)\ |\vec{x}; m\rangle$ with $|x, m\rangle = |x\rangle \otimes |m\rangle$

can be represented $\begin{pmatrix} \psi_\uparrow(\vec{x}) \\ \psi_\downarrow(\vec{x}) \end{pmatrix}$

Now cute fact — magnetic moment of electron proportional to spin

- spinning charged objects have mag moment
- $\vec{\mu}$ is a vector
- only vector in problem is $\vec{S}$

ergo $\vec{\mu} = \gamma \vec{S}$

proportionality constant "gyro-magnetic ratio"

Now how big is γ

Dynamics:

$$H = -\gamma \vec{B} \cdot \vec{S}$$

breaks rotation symmetry
all direction <u>not</u> the same

First let us solve time ind.
schrodinger equation for spin ½ particle
in const. external field

say $\vec{B} = B_0 \hat{z}$

$$H = -\gamma B_0 \hat{z} \cdot \hat{S} = -\gamma B_0 \hat{S}_z$$

eigenstates of H are eigenstates of
$\hat{S}_z$ $|\uparrow\rangle$ and $|\downarrow\rangle$

$H|\uparrow\rangle = -\gamma B_0 \frac{\hbar}{2} |\uparrow\rangle$ $\qquad E_\uparrow = -\frac{\gamma B_0 \hbar}{2}$

$H|\downarrow\rangle = -\gamma B_0 (-\frac{\hbar}{2}) |\uparrow\rangle$ $\qquad E_\downarrow =$

$\Delta E = E_\downarrow - E_\uparrow = +\gamma B_0 \hbar$

suppose the electron is in $|\downarrow\rangle$ and makes a
transition to $|\uparrow\rangle$ what frequency photon is

emmited $\quad w_0 = \frac{|\Delta E|}{\hbar} = \gamma B_0$

guts of NMR or ESR

Now suppose that at $t=0$ system is not in an eigenstate what is time evolution

general form

$$|\Psi(t)\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle \qquad |a|^2+|b|^2=1$$

schrodinger equation

$$H|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle$$

$$|\Psi(t)\rangle = e^{-iE_\uparrow t/\hbar}a(0)|\uparrow\rangle + e^{-iE_\downarrow t/\hbar}b(0)|\downarrow\rangle$$

$$= e^{+i\frac{\gamma B_0 t}{2}}a_0|\uparrow\rangle + e^{-i\frac{\gamma B_0}{2}t}b(0)|\downarrow\rangle$$

simple vector representation

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} e^{i\frac{\gamma B_0}{2}t}a(0) \\ e^{-i\frac{\gamma B_0}{2}t}b(0) \end{pmatrix}$$

Now what does this mean

let us look at expection values of $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in $|\Psi(t)\rangle$

$$\frac{\hbar}{2}\sigma_z$$

$$\langle\Psi(t)|\,\hat{S}_z\,|\Psi(t)\rangle = (a^*(t) \quad b^*(t)) \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a^*(t)\,a(t) - b^*(t)\,b(t)\right)$$

$$= \frac{\hbar}{2}\left(|a(0)|^2 - |b(0)|^2\right)$$

it is time independent!

why?

recall $\frac{d}{dt}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$

Now if $[\hat{H},\hat{A}]=0$ then $\langle\hat{A}\rangle$ is time independent

but $[\hat{H},\hat{S}_z]=0$ since $\hat{H} = -\gamma B_0 \hat{S}_z$

so $\frac{d\langle S_z\rangle}{dt}=0$

what about

$\langle S_x\rangle$ Now $\langle S_x\rangle = (a^* \; b^*)\begin{pmatrix} 0 & \hbar/2 \\ \frac{\hbar}{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$$= \frac{\hbar}{2}\left(a^* b + b^* a\right)$$

but $a(t) = e^{i\frac{\gamma B_0}{2}t}\, a(0)$ $\quad b(t) = e^{-i\frac{\gamma B_0 t}{2}}\, b(0)$

$a^*(t) = e^{-i\frac{\gamma B_0 t}{2}}\, a^*(0)$ $\quad b^*(t) = e^{+i\frac{\gamma B_0 t}{2}}\, b^*(0)$

so

$$\langle S_x \rangle = \frac{\hbar}{2}\left(a^* b + b^* a\right) = \frac{\hbar}{2}\left[a^*(0)\, e^{-\frac{i\gamma B_0 t}{2}}\, b(0)\, e^{-\frac{i\gamma B_0 t}{2}} + b^*(0)\, e^{\frac{i\gamma B_0 t}{2}}\, a(0)\, e^{+\frac{i\gamma B_0 t}{2}}\right]$$

writing $\quad a(0) = e^{+i\delta_\alpha}\, |a(0)|$

$\quad b(0) = e^{+i\delta_\beta}\, |b(0)|$

$$\langle S_x \rangle = \frac{\hbar}{2}\, |a(0)|\,|b(0)| \left[e^{i(\delta_\alpha - \delta_\beta)}\, e^{-i\gamma B_0 t} + e^{+i(\delta_\alpha - \delta_\beta)}\, e^{+i\gamma B_0 t}\right]$$

~~$\frac{\hbar}{2}\, |a(0)|\,|b(0)|\, \cos$~~

$$= \frac{\hbar}{2}\, |a(0)|\,|b(0)|\; 2\cos\left(\gamma B_0 t + \underbrace{\delta_\alpha - \delta_\beta}_{\text{phase shift}}\right)$$

I can always pick $\delta_\alpha = \delta_\beta = 0$ by fixing $t=0$ point and arbitrary phase

$$\langle S_x \rangle = \hbar\, |a(0)|\,|b(0)|\, \cos(\omega t) \qquad \omega = \gamma B_0$$

Now what about

$|a(0)|\,|b(0)|$

Note $|a(0)|^2 + |b(0)|^2 = 1$

I can always parameterize them as

$|a(0)| = \cos(\alpha/2) \qquad \pi > \alpha > 0$

$|b(0)| = \sin(\alpha/2)$

$|a(0)|\,|b(0)| = \cos(\alpha/2)\sin(\alpha/2) = \frac{1}{2}\sin(\alpha)$

so $\langle S_x \rangle = \frac{\hbar}{2}\sin(\alpha)\cos(\gamma B_0 t)$

similarly $\langle S_y \rangle = \frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t)$

while $\langle S_z \rangle = \frac{\hbar}{2}\left[|a(0)|^2 - |b(0)|^2\right] = \frac{\hbar}{2}\left(\cos^2(\tfrac{\alpha}{2}) - \sin^2(\tfrac{\alpha}{2})\right)$

$= \frac{\hbar}{2}\cos(\alpha)$

- Note α says how "aligned" spin is to z axis

- spin precesses about z axis of expectation
- precession frequency

$$\omega = \gamma B_0 = \frac{g \mu_0}{\hbar} B_0$$

is exactly $\frac{\Delta E}{\hbar}$ where $\Delta E = E_\uparrow - E_\downarrow$

- suppose $\alpha = 0$ $\langle S_z \rangle = 0$

then at $t=0$ $\langle S_x \rangle = \frac{\hbar}{2}$ 100% of measurments of S_x give $+\frac{\hbar}{2}$

at $\pi = \gamma B t$ $\langle S_x \rangle = -\frac{\hbar}{2}$ 100% of measurement of S_x give $-\frac{\hbar}{2}$

at $\frac{\pi}{2} = \gamma B t$ $\langle S_y \rangle = \frac{\hbar}{2}$ etc.

- Align system in $|\uparrow\rangle$ in $\hat{x}$ and it remains $|\uparrow\rangle$ somewhere in $\hat{x}-\hat{y}$ plane but the line about which it is up processes

- align system in $|\uparrow\rangle$ along some axis in the $\hat{x}-\hat{z}$ plane
- and it remains $|\uparrow\rangle$ but in a line pressing about z axis

Addition of angular momentum

example:

$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ 1- particle

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ 2- particles

$\hat{\vec{J}} = \hat{\vec{J}}^1 + \hat{\vec{J}}^2$ generic

Now if 1 & 2 are different systems

$[\hat{J}_i^{(1)}, \hat{J}_j^{(2)}] = 0$

Consider $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$

so $[\hat{J}_i, \hat{J}_j] = [\hat{J}_i^{(1)}, \hat{J}_j^{(1)}] + \cancel{[\hat{J}_i^{(1)}, \hat{J}_j^{(2)}]}$

$+ \cancel{[\hat{J}_i^{(2)}, \hat{J}_j^{(1)}]} + [\hat{J}_i^{(2)}, \hat{J}_j^{(2)}]$

$= i\,\varepsilon_{ijk}\,\hat{J}_k^{(1)} + i\,\varepsilon_{ijk}\,\hat{J}_k^{(2)}$

$= i\,\varepsilon_{ijk}\,[\hat{J}_k^{(1)} + \hat{J}_k^{(2)}] = i\,\varepsilon_{ijk}\,\hat{J}_k$

ang. mo. commutation relation

the sum of two ang. mo. is an ang. mo!

Now I can have two basis for writing states

$\cancel{|J_1 J_2\rangle}$ $|J_1\ J_2\ m_1\ m_2\rangle = |J_1\ m_1\rangle \otimes |J_2\ m_2\rangle$

or $|J\ m\rangle$ total J total

clearly I can go between these with a unitary trans

Simple example

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ $\qquad S_1 = \frac{1}{2} \qquad S_2 = \frac{1}{2}$

in $|s_1, s_2, m_1\ m_2\rangle$ basis 4 states

$|\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\rangle \equiv |\uparrow\uparrow\rangle$ $\qquad |\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\rangle \equiv |\uparrow\downarrow\rangle$ $\qquad |\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\rangle \equiv |\downarrow\uparrow\rangle$

$|\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\rangle \equiv |\downarrow\downarrow\rangle$

let's construct states with good S, m_s

- start with $|\uparrow\uparrow\rangle$

this state has $m_1 = +\frac{1}{2}$ $m_2 = +\frac{1}{2}$

$$\hat{S}_z |\uparrow\uparrow\rangle = \hat{S}_z^{(1)} + \hat{S}_z^{(2)} |\uparrow\uparrow\rangle = S_z^1 |\uparrow\rangle \otimes S_z^2 |\uparrow\rangle$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) \hbar |\uparrow\uparrow\rangle$$

$$= \hbar |\uparrow\uparrow\rangle$$

so $|\uparrow\uparrow\rangle$ has $S_z = \hbar$

what about $\hat{S}_+ |\uparrow\uparrow\rangle = \hat{S}_+^{(1)} + \hat{S}_+^{(2)} |\uparrow\uparrow\rangle$

$$= (\hat{S}_+^1 |\uparrow\rangle) \otimes \hat{S}_+^{(2)} |\uparrow\rangle$$

$$= 0 * 0 = 0$$

$\hat{S}_+$ annihilates $|\uparrow\uparrow\rangle$ so $|\uparrow\uparrow\rangle$ is state with max m thus this state has $S = 1$

$$|\uparrow\uparrow\rangle = |S=1, m=1\rangle$$

$$S_- |S=1, m=1\rangle = \hbar \sqrt{S(S+1) - m(m-1)}\; |S=1, m=0\rangle$$

$$= \hbar \sqrt{2}\; |S=1, m=0\rangle$$

$$|S=1, m=0\rangle = \frac{1}{\sqrt{2}\hbar} \hat{S}_- |S=1\ m=1\rangle$$

$$= \frac{1}{\sqrt{2}\hbar} \left(\hat{S}_-^{(1)} + \hat{S}_-^{(2)}\right) |\uparrow\uparrow\rangle$$

$$= \frac{1}{\sqrt{2}\hbar} \left[\hat{S}_-^{(1)} |\uparrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes \hat{S}_-^{(2)} |\uparrow\rangle\right.$$

now $S_- |\uparrow\rangle = \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\ |\downarrow\rangle = \hbar|\downarrow\rangle$

$$|S=1, m=0\rangle = \frac{1}{\sqrt{2}} \left(|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \right)$$

$|S=1\ m=-1\rangle = |\downarrow\downarrow\rangle$ by analogy to $|\uparrow\uparrow\rangle$

- three states $|S=1\ m=1\rangle$ $|S=1\ m=0\rangle$ $|S=1\ m=-1\rangle$

but original basis has 4 what is missing state

- must orthogonal to other three
- must have $m=0$ (note 2 of 4 states had $m=0$ but only one above

easy to see last state

$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

clearly $\langle\Psi| s=1\ m=0\rangle = 0$

$S_z|\Psi\rangle = 0$

what is s for $|\Psi\rangle$

claim $s=0$: proof

$$S_+|\Psi\rangle = (\hat{S}_+^{(1)} + \hat{S}_+^{(2)})|\Psi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(S_+^{(1)}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + S_+^{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left((0 - |\uparrow\uparrow\rangle) + (|\uparrow\uparrow\rangle - 0)\right) = 0$$

$m=0$ is max m $\therefore$ $s=0$

two basis sets

$|s=0\ m=0\rangle$ $\quad |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

$|s=1\ m=1\rangle$

$|s=1\ m=0\rangle$

$|s=1\ m=-1\rangle$

Coefficients to go from 1 basis to other

$$|s\ M\rangle = \sum_{m_1 m_2} \underbrace{\langle s_1\, s_2\, m_1\, m_2 | s\ M\rangle}_{\text{called Clebsch-Gordon coefficients}} |s_1\, s_2\, m_1\, m_2\rangle$$

we have just computed some

$$|s=1\ m=1\rangle = |\uparrow\uparrow\rangle \equiv |\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

so $\boxed{\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} | 1\ 1\rangle = 1}$

$$|s=1\ m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\ \tfrac{1}{2}\rangle$$

so

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}} | 1\ 0\rangle = \frac{1}{\sqrt{2}}$$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\ \tfrac{1}{2} | 1\ 0\rangle = \frac{1}{\sqrt{2}}$$

$$|s=0 \quad m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}|0\ 0\rangle = \frac{1}{\sqrt{2}}$$

$$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\ \tfrac{1}{2}|0\ 0\rangle = \frac{-1}{\sqrt{2}}$$

etc.

bad news . a pain to compute

good news : only need to compute once
and then make a table !

Seems ~~mostly~~ like pure math
why bother

- Consider a spin - spin interaction
  eg hyperfine in atomic physics

$$H = -\alpha\, \vec{S}_1 \cdot \vec{S}_2$$ mag moment - mag moment interaction

- rotationally invariant

now what is $\vec{S}_1 \cdot \vec{S}_2 \Rightarrow$ $\vec{S}\cdot\vec{S} = (\vec{S}_1+\vec{S}_2)\cdot(\vec{S}_1+\vec{S}_2)$

$$= \vec{S}_1\cdot\vec{S}_1 + \vec{S}_2\cdot\vec{S}_2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}\left[S^2 - \vec{S}_1^{\,2} - S_2^2\right]$$

$$= \frac{\hbar^2}{2}\left[s(s+1) - s_1(s_1+1) - s_2(s_2+1)\right]$$

if $\quad s_1 = \frac{1}{2} \qquad s_2 = \frac{1}{2}$

$$= \frac{\hbar^2}{2}\left[s(s+1) - \frac{3}{2}\right]$$

two cases $\qquad s=0 \qquad s=1$

$s=0 \qquad \vec{S}_1\cdot\vec{S}_2 = -\frac{3}{4}\hbar^2$

$s=1 \qquad \vec{S}_1\cdot\vec{S}_2 = +\frac{1}{4}\hbar^2$

$\hat{H}\,|s=0\rangle = -\frac{3}{4}\hbar^2\alpha$

$$\left.\begin{array}{l}\hat{H}\,|s=0, m=0\rangle = +\frac{1}{4}\hbar^2\alpha \\ \hat{H}\,|s=0, m=1\rangle = \frac{1}{4}\hbar^2\alpha \\ \hat{H}\,|s=0, m=-1\rangle = \frac{1}{4}\hbar^2\alpha\end{array}\right\}\ \text{degenerate}$$

More generally

$s_1 = \frac{1}{2}$ $s_2 = \frac{1}{2}$ $\Rightarrow$ $S = 1, 0$

General Case

s_1, s_2 $S =$ ~~S_{min}~~ $S_{max}, S_{max}-1, S_{max}-2, \ldots, S_{min}$

$S_{max} = S_1 + S_2$ proof look at m

$S_{max} = M_{max} = m_{1\,max} + m_{2\,max} = S_1 + S_2$

claim: $S_{min} = |S_1 - S_2|$

proof: bit complicated I'll just do numerology.

total # of states: $(2S_1+1)\times(2S_2+1) = 4S_1S_2 + 2(S_1+S_2) + 1$

$$= \sum_{S=S_{min}}^{S_{max}} (2S+1) = 2\sum_{S=S_{min}}^{S_{max}} S + \sum_{S_{min}}^{S_{max}} 1$$

$$= 2\left[\frac{S_{max}(S_{max}+1)}{2} - \frac{S_{min}(S_{min}+1)}{2}\right] + (S_{max}-S_{min})$$

$$= S_{max}^2 + S_{max} - S_{min}^2 - S_{min} + S_{max} - S_{min}$$

$$= (S_1+S_2)^2 + 2(S_1+S_2) - S_{min}^2 - S_{min}$$

suppose $S_{min} = S_1 \pm S_2$ $S_{min}^2 = S_1^2 + S_2^2 - 2S_1 -$

$s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2} \Rightarrow s = 1, 0$ all others not allowed

General Case

$J = \vec{J}_1 + \vec{J}_2 \Rightarrow J = J_{max}, J_{max} - 1, \cdots J_{min}$

$J_{max} = J_1 + J_2$ proof look at $M = M_1 + M_2$

$J_{max} = M_{max} = M_{1\,max} + M_{2\,max} = J_1 + J_2$ Q.E.D.

$J_{min} = |J_1 - J_2|$

proof is a bit complicated

we'll do numerology

consider $J_1 = \frac{1}{2}$ $J_2 = L$ $L > 0$

\# of states $2(2L+1) = 4L+2$

$J_{max} = L + \frac{1}{2}$ \# of states $2(L+\frac{1}{2})+1 = 2L+2$

$J_{min} = L - \frac{1}{2}$ \# of states $2(L-\frac{1}{2})+1 = 2L$

$4L+2$ ✓

~~consider $J_1 = 2$ $J_2 = 3$~~

~~\# of states = $(2(2)+1) \times (2(3)+1) = 35$~~

~~$J_{max} = 5$ \# of states $2(5)+1 = 11$~~

~~$J = 4$ \# of states $2(4)+1 = 9$~~

~~$J = 3$ \# of states $2(3)+1 = 7$~~

~~$J = 2$ \# of states $2(2)+1 = 5$~~

~~$J = 1$ \# of states $2(1)+1 = 3$~~

consider $J_1 = 2$ $J_2 = 3$

total # of states $(2(2)+1) \times (2(3)+1) = 35$

$J_{max} = 5$	# of states $2(5)+1 = 11$.
$J = 4$	$2(4)+1 = 9$
$J = 3$	$2(3)+1$ 7
$J = 2$	$2(2)+1$ 5
$J_{min} = 1$	$2(1)+1$ 3
	35 ✓

~~what possible spins do I get if~~

tables to find

$\langle J_1 \; J_2 \; m_1 \; m_2 | J \; m \rangle$

Quantum mechanics of two or more particles

basis: (ignoring spin

$|\vec{r}_1, \vec{r}_2\rangle$

$\langle \vec{r}_1, \vec{r}_2 | \Psi\rangle = \Psi(\vec{r}_1, \vec{r}_2)$ interp. $P(\vec{r}_1, \vec{r}_2) = \Psi^*(\vec{r}_1, r_2)\Psi(r_1, r_2)$

$\int d\vec{r}_1\, d\vec{r}_2\, \Psi^*(\vec{r}_1, \vec{r}_2)\,\Psi(r_1, r_2) = 1$

General: $H = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(\vec{r}_1, \vec{r}_2)$

$H|\Psi\rangle = i\hbar \frac{d|\Psi\rangle}{dt}$

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2)\right]\Psi(r_1, r_2) = E\,\Psi(\vec{r}_1, r_2)$$

for an energy Eigenstate

$$\Psi(\vec{r}_1 \vec{r}_2, t) = e^{-iEt/\hbar}\,\Psi(\vec{r}_1, \vec{r}_2; t=0)$$

useful transformation:

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ $\qquad \vec{r} = \vec{r}_1 - \vec{r}_2$

cm $\qquad$ relative

simple algebra (Homework)
gives

$$\left[-\frac{\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2 + u(r,R) \right] \Psi(r,R) = E\, \Psi(r,R)$$

where $u(r,R) \equiv V(r_1, r_2)$

$M = m_1 + m_2$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

interesting case:

suppose $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$ e.g.?

or $u(r,R) = u(r)$

translationally invariant

- system separates

$$\Psi(r,R) = \phi_r(r)\, \Phi_R(R)$$

$$\left[-\frac{\hbar^2}{2m} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2 + u(r) \right] \phi_r(r) \Phi_R(R) = E\, \phi_r(r) \Phi_R(R)$$

divide both sides by $\phi_r(r) \Phi_R(R)$

$$\frac{\frac{-\hbar^2}{2m}\nabla_R^2\,\Phi_R(R)}{\Phi_R(R)} + \frac{\left[\frac{-\hbar^2}{2m}\nabla_r^2 + U(r) - E\right]\phi_r(r)}{\phi_r(r)} = 0$$

each term must be constant

$$\frac{-\hbar^2}{2m}\nabla_R^2\,\Phi_R = \underset{\text{name this } \frac{P^2}{2m}}{\text{const}}\;\Phi_R$$

c.m. momentum

$$\Phi_R = \frac{e^{-i\vec{P}\cdot R}}{\sqrt{(2\pi)^3}}$$

convention ~~use~~ not renormalizable

$$\frac{\left(\frac{-\hbar^2}{2m}\nabla_r^2 + U(r) - E\right)\phi_r(r)}{\phi_r(r)} = -\frac{P^2}{2m}$$

or

$$\left[\frac{-\hbar^2}{2m}\nabla_r^2 + U(r)\right]\phi_r(r) = \left(E - P^2/2m\right)\phi_r(r)$$

$$\underset{\text{Internal}}{|||}$$

[illegible]

got an effective 1-body problem with

$$m \to \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$E \to E - \frac{P^2}{2m} = E_{\text{internal}}$$

e.g. hydrogen atom including proton

total energy is internal energy + $\frac{P^2}{2m}$

internal energy $\equiv \frac{\tilde{E}_1}{n^2}$

but $\tilde{E}_1$ is E_1 with $m \to \mu$

recall $E_1 = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$

so $\tilde{E}_1 = \frac{\mu}{m} E_1$

Note that $\mu = \frac{m_1 m_2}{m_1+m_2}$

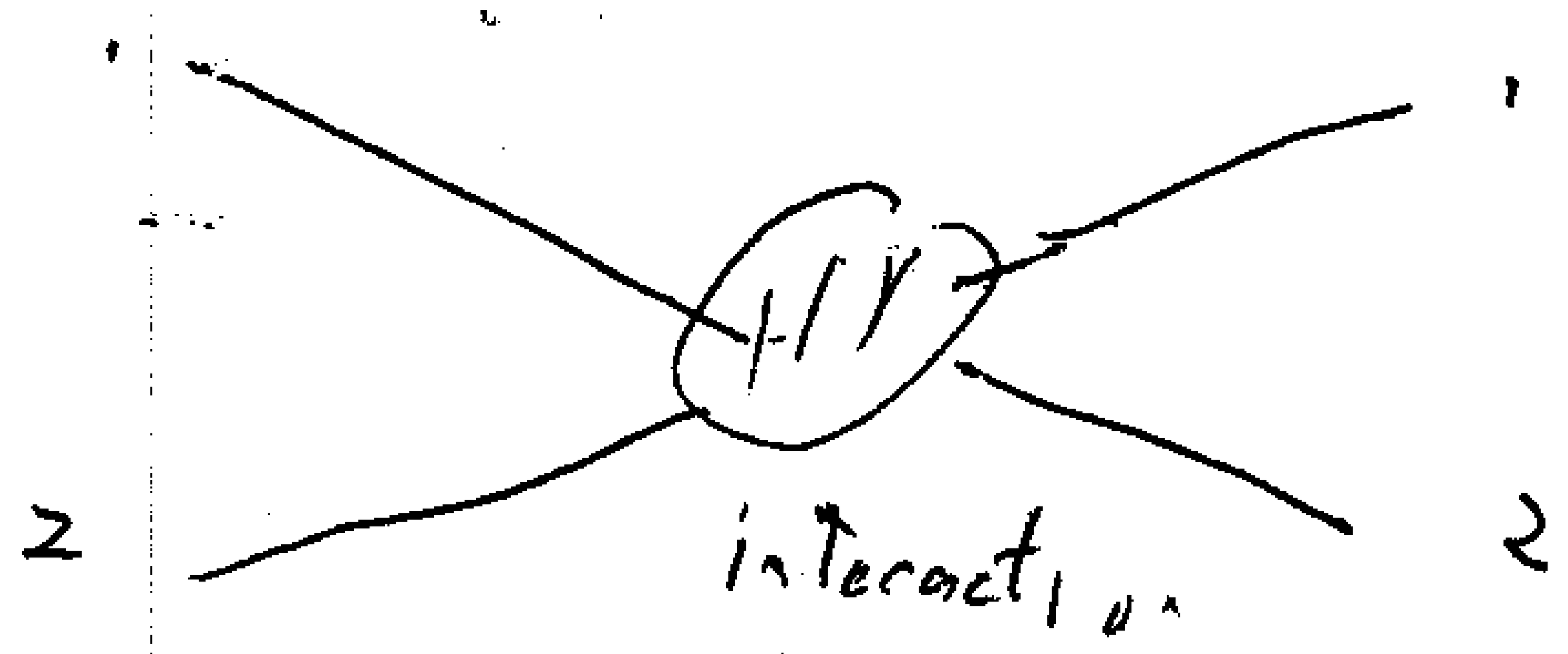
now if $m_2 \gg m_1$ $\mu \approx \frac{m_1 m_2}{m_2(1+m_1/m_2)} = m_1\left(1-\frac{m_1}{m_2}\right)$

so if $m_2/m_1 \to \infty$ $\mu \to m_1$ $m_p/m_e \sim 2000$

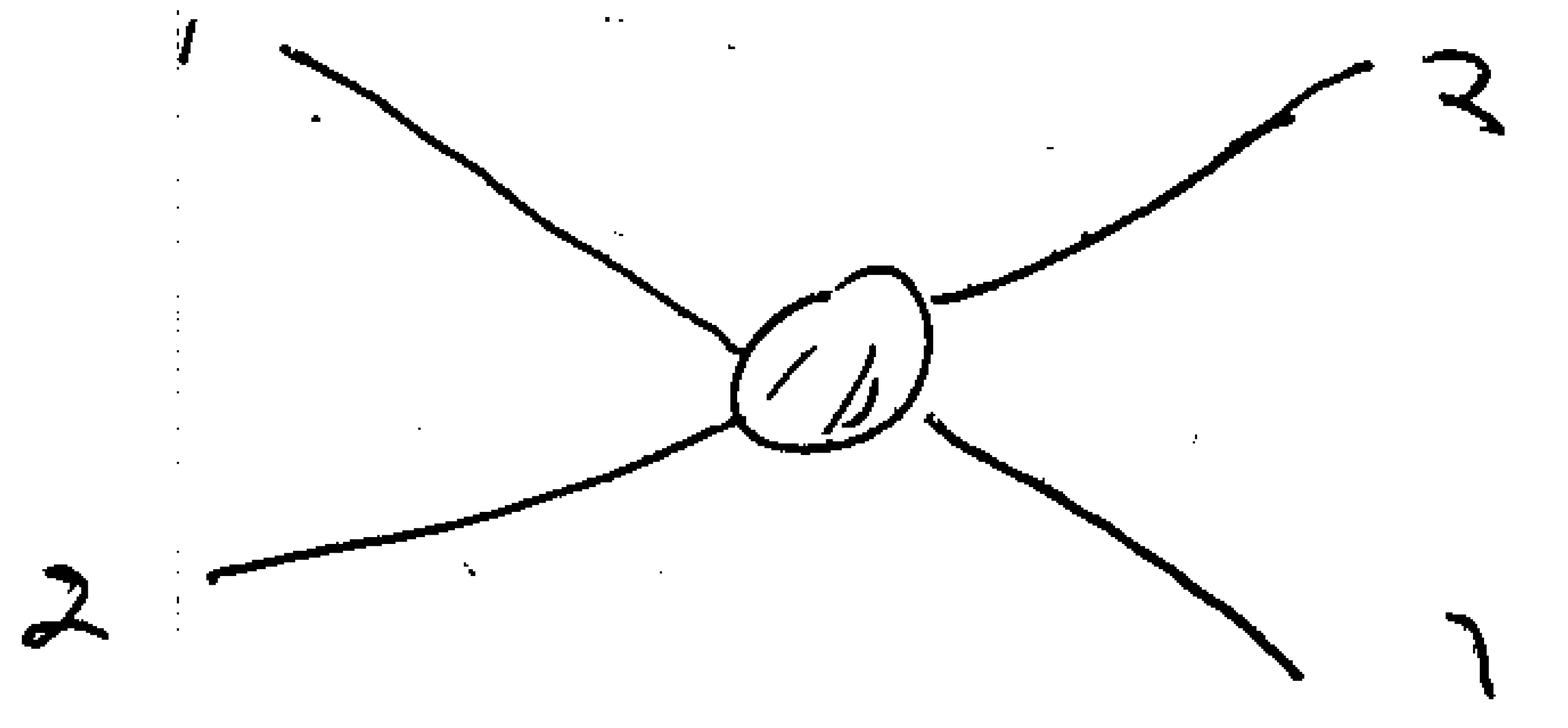
- Identical vs distinguishable particles

  - classically all particles are distinguishable
    just follow it or put a spot on it
  - quantum more subtle
    - can't follow it (uncertainty
    - can't paint a spot

experimental question - suppose I scatter to electrons (both of which are spin up so I can't use spin to tell appart)



or



where 1 & 2 label which electron

three questions:

1) can I tell these two appart?

2) If not, are they same process with same scattering quantum state or are the different

3) Does the answer to 2) matter in physical observables or is it only for the philosophy dept.

Answers:

1) No! How could all electrons "look" alike (no hair theorem)

2) this question cannot be answered by pure thought. Both answers are logically possible.

3) The answer to 2) <u>does</u> matter experimentally and we can appeal to experiment to answer it

Consider for simplicity a system with two particles (no spin or spin all in same state) with particles identical

$$|\vec{r}_1, \vec{r}_2\rangle = |r_1\rangle \otimes |r_2\rangle$$

label

. basis states where 1 & 2 label which particle if particles can be distinguished

$$|\vec{r}_a, \vec{r}_b\rangle \neq |\vec{r}_b, \vec{r}_a\rangle$$

i.e. system with the first particle at point a & 2nd at b is different from 1st particle at a and second at b.

Consider an exchange operator (switches particles

$$\hat{P}\,|\vec{r}_a, \vec{r}_b\rangle = |\vec{r}_b, \vec{r}_a\rangle$$

if particles are identical and if the states were different (question 2 ~~if~~) then

$$P\,|\vec{r}_a, \vec{r}_b\rangle \neq |\vec{r}_b, \vec{r}_a\rangle$$

but if the same in sense of question 2

then

$|\vec{r}_a, \vec{r}_b\rangle$ is physically equivalent $|\vec{r}_b, \vec{r}_a\rangle$

does physically equivalent mean equal?

not necessarily! phases cannot be directly measured

so $\hat{P}\,|\vec{r}_a, \vec{r}_b\rangle = e^{i\phi}\,|\vec{r}_b, \vec{r}_a\rangle$

can we fix ϕ ?

consider

$$\hat{P}^2|\vec{r}_a, \vec{r}_b\rangle = \hat{P}\,\hat{P}\,|\vec{r}_a, \vec{r}_b\rangle = \hat{P}\,e^{i\phi}\,|\vec{r}_b, \vec{r}_a\rangle$$
$$= e^{i\phi}\,\hat{P}\,|\vec{r}_b, \vec{r}_a\rangle = e^{2i\phi}\,|\vec{r}_a, \vec{r}_b\rangle$$

but clearly $\hat{P}^2 = \mathbb{1}$ switching & switching back brings you to same state so

$$\hat{P}^2|\vec{r}_a, \vec{r}_b\rangle = |\vec{r}_a, \vec{r}_b\rangle \quad \text{or} \quad e^{2i\phi} = 1$$

two solutions $e^{i\phi} = \pm 1$ so $\hat{P}|\vec{r}_a, \vec{r}_b\rangle = +|\vec{r}_b, \vec{r}_a\rangle$ or $\hat{P}|\vec{r}_a, \vec{r}_b\rangle = -|\vec{r}_b, \vec{r}_a\rangle$

Now it happens nature <u>does</u> behave this way (experimentally)

some particles behave with +

these are boson (from Bose)

other particle behave with −

these are fermions

Note this constains wavefunctions

$$\Psi(\vec{r}_1, \vec{r}_2) = \langle \vec{r}_1, \vec{r}_2 | \Psi \rangle$$

for bosons

$$\Psi(\vec{r}_1, \vec{r}_2) = \langle \vec{r}_1, \vec{r}_2 | \Psi \rangle = \langle \vec{r}_2, \vec{r}_1 | \Psi \rangle$$

$$= \Psi(\vec{r}_2, \vec{r}_1)$$

switch labels of particles get same wave function

for fermions

$$\Psi(\vec{r}_1, \vec{r}_2) = \langle \vec{r}_1, \vec{r}_2 | \Psi \rangle = \langle \vec{r}_2, \vec{r}_1 | (-1) | \Psi \rangle = -\langle \vec{r}_2, \vec{r}_1 | \Psi \rangle$$

$$= -\Psi(r_1, r_2)$$

comments

- integral spin particles are boson
- $\frac{1}{2}$ integral spin particles are fermions

(spin - statics connection)

- true empirically
- derivable from causality & unitarity in quantum

Simple examples — "non-interacting" particles
particle are each in an external potential
but we can neglect potential between
them

single particle eigenstates

$\psi_a(\vec{r})$, $\psi_b(\vec{r})$ ...

two particle states:

1) distinguishable $\psi_{ab}(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\, \psi_b(\vec{r}_2)$
particle 1 is in a
particle 2 is in b

normalization factor (h.w.)

2) bosons $\psi_{ab}(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right]$

3) fermions $\psi_{ab}(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right]$

what if $a = b$ case 1 & 2 o.k. but case 3 gives
zero!

You cannot put two fermions in same
single particle level!

this is the Pauli principle (basis of all chemistry)

Odd consequences I have two particles restricted to be in 1 of 2 single particle states call them H or T

suppose further that all configurations are equally likely (zero temp)

case 1) distinguishable 4 states: $|HH\rangle$ $|HT\rangle$ $|TH\rangle$ $|TT\rangle$

if I randomly select config $\frac{1}{2}$ the time both particles are in same ~~state~~ single particle state

2) fermions 1 state: $\frac{1}{\sqrt{2}}\left(|HT\rangle - |TH\rangle\right)$ is only possible state

I never will have both particles in same single particle state

3) boson 3 states: $|HH\rangle$ $\frac{1}{\sqrt{2}}\left(|HT\rangle + |TH\rangle\right)$ $|TT\rangle$

$\frac{2}{3}$ of configurations have both particles in same state

fermions are anti social
bosons are social

Cute case worked out in book demonstrates

2 levels a, b $a \neq b$

Two particles in these levels

fermions $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|a,b\rangle - |b,a\rangle\right)$

bosons $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|ab\rangle + |b,a\rangle\right)$

consider some quantity like $\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - \langle 2 x_1 x_2 \rangle$ which tells us how close the particles like to be to each other

fermion case $\langle \Psi | \hat{x}_1^2 | \Psi \rangle = \frac{1}{\sqrt{2}} \left(\langle a b| - \langle ba| \right) | \hat{x}_1^2 | \frac{1}{\sqrt{2}} \left(|ab\rangle - |ba\rangle \right)$

$= \frac{1}{2}\left[\langle a | \hat{x}^2 | a\rangle + \langle b | \hat{x}^2 | b \rangle\right]$ cross terms vanish

$\langle \Psi | \hat{x}_2^2 | \Psi \rangle = \frac{1}{2}\left[\langle a | \hat{x}^2 | a\rangle + \langle b | \hat{x}^2 | b \rangle\right]$

what about $\langle \Psi | \hat{x}_1 \hat{x}_2 | \Psi \rangle$

$= \frac{1}{\sqrt{2}} \left(\langle a b| - \langle ba| \right) \hat{x}_1 \hat{x}_2 \frac{1}{\sqrt{2}} \left(|ab\rangle - |ba\rangle \right)$

$= \frac{1}{2} \left\{ \langle ab | \hat{x}_1 \hat{x}_2 | ab \rangle - \langle ab | \hat{x}_1 \hat{x}_2 | ba \rangle - \langle ba | x_1 x_2 | ab \rangle + \langle ba | x_1 x_2 | ba \rangle \right\}$

$= \frac{1}{2} \langle a|\hat{x}|a\rangle\langle b|\hat{x}|b\rangle - \langle a|\hat{x}|b\rangle\langle b|\hat{x}|a\rangle - \langle b|\hat{x}|a\rangle\langle a|\hat{x}|b\rangle + \langle b|\hat{x}|b\rangle\langle a|\hat{x}|a\rangle$

$= \langle a|\hat{x}|a\rangle \langle b|\hat{x}|b\rangle - |\langle b|\hat{x}|a\rangle|^2$

PV

or

$$\langle\psi|(\hat{x}_1-\hat{x}_2)^2|\psi\rangle = \langle a|\hat{x}^2|a\rangle + \langle b|\hat{x}^2|b\rangle - 2\langle a|\hat{x}|a\rangle\langle b|\hat{x}|b\rangle + 2|\langle b|\hat{x}|a\rangle|^2$$

fermion case

boson case

$$\langle\psi|(\hat{x}_1-\hat{x}_2)^2|\psi\rangle = \langle a|\hat{x}^2|a\rangle + \langle b|\hat{x}^2|b\rangle - 2\langle a|\hat{x}|a\rangle\langle b|\hat{x}|b\rangle - 2|\langle b|\hat{x}|a\rangle|^2$$

opposite sign (same calc)

$$\langle(\hat{x}_1-\hat{x}_2)^2\rangle_{\text{fermion}} - \langle(\hat{x}_1-\hat{x}_2)^2\rangle_{\text{boson}}$$

$$= 4|\langle b|\hat{x}|a\rangle|^2 \geq 0$$

the fermions are "pushed appart" relative to the bosons

but note there was no force between them in Hamiltonian

Non interacting particles interact thru exchange

- when $|\langle b|\hat{x}|a\rangle|^2 \ll \langle a|\hat{x}^2|a\rangle$ exchange doesn't matter

— the real case

- fermions have spin as well as space
- bosons can have spin

— full state has spinor and spatial wavefunction

Exchange ($\hat{P}$) means exchange everthing not just space

eg basis states for single spin $\frac{1}{2}$

particle state $|r_0\ m_0\rangle$

2 particle states (distinguishable)

$$|r_1\ m_1\ \ r_2 m_2\rangle$$

Now $P|r_1 m_1\ r_2 m_2\rangle = |r_2\ m_2\ \ r_1\ m_1\rangle$

if identical $|r_1\ m_1\ r_2\ m_2\rangle = -|r_2\ m_2\ r_1\ m_1\rangle$

what is $|r_1\ m_2\ r_2\ m_1\rangle$ not imediately related to above unless $m_1 = m_2$

eg. 2 spin $\frac{1}{2}$ particles

$\psi_a \chi_a$ $\psi_b \chi_b$

space spin space spin

Combine $\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]\left[\chi_a^{(1)}\chi_b^{(2)} \mp \chi_b^{1}\chi_a^{2}\right]$

space - spin (sym - anti) or anti - sym

more generally $\left[\psi(r_1, r_2) \pm \psi(r_2, r_1)\right]\left[\chi_a^1 \chi_b^2 \mp \chi_b^1 \chi_a^2\right]$

space spin

Note if space is symmetric then spin anti sym
but anti-sym means total $|S=0\rangle$

if space is anti-symmetric then spin sym
which means total spin $|S=1\rangle$

Consider He atom : two electrons
if two electrons are $|S=0\rangle$ (para helium)
has symmetric space wave functions
(both electrons can be in same state
spatially