QUANTUM PHYSICS II

PROBLEM SET 7

due November 12th, before class

I. BORN RULE

A spinless particle moving in one dimension is described by the state $|\psi\rangle = \int_{-\infty}^{\infty} dx \sqrt{42} e^{-42|x|} |x\rangle$, where $|x\rangle$ are the eigenstates of position with eigenvalue x.

i) What is the probability of finding the particle between $x = 0$ and $x = 17$?

ii) If the momentum is measured, what are the possible outcomes and their respective probabilities?

II. DEGENERATE MATTERS

Consider a three-dimensional harmonic oscillator with the hamiltonian

$$\hat{H}_0 = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2). \tag{1}$$

1) Show that the eigenstates of $\hat{H}_0$ are of the form $|n_x, n_y, n_z\rangle = |n_x\rangle|n_y\rangle|n_z\rangle$, where $|n\rangle$ are the eigenstates of the one-dimensional harmonic oscillator. Find the energy of these states and show that the first excited state is degenerate. Can you understand this degeneracy on the ground of any symmetry?

2) An extra potential of the form $V(x, y, z) = \lambda\delta(x)$ is added to $\hat{H}_0$. Find the shift in energy of the ground state and the first excited states correct up to order $\mathcal{O}(\lambda)$.

III. DON'T FORGET YOU BRAS AND KETS

$$\langle y\,|\hat{p}|x\rangle = \tag{2}$$

$$\langle y\,|\hat{p}|\psi\rangle = \tag{3}$$

$$\langle y\,|\hat{p}^2|\psi\rangle = \tag{4}$$

$$\langle y\,|f(\hat{p})|\psi\rangle = \tag{5}$$

where $|x\rangle, |x\rangle$ are eigenkets of the position operator, $|\psi\rangle$ and arbitrary ket and $f(\hat{p})$ an arbitrary function of the momentum operators (which you can assume has a nice Taylor expansion). You should write your answer in terms of $\psi(x) = \langle x\,|\psi\rangle$ where appropriate.

IV. THE GREAT VARIATIONAL CHALLENGE

Make a variational calculation of the ground state energy of anharmonic oscillattor with hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \lambda\hat{x}^4. \tag{6}$$

First use dimensional analysis to determine the form of the answer up to a numerical constant. Then devise your own variational ansätz to estimate the numerical constant. Your ansätz may contain any number of parameters. The person with the smallest energy will win a doughnut and a certificate of "Grand Master of All Variational Caculations". You are allowed to use a computer for analytic or numerical steps. This is a no-holds barred competition and only results matter.

V. OPTIONAL: DON'T FORGET YOUR BRACKETOLOGY

Consider a spinless particle moving in one-dimension. The eigenbasis of the position operator $|x\rangle$ is indexed by the values of the position x. Write the following expressions in terms of the position space wave functions $\psi(x) = \langle x|\psi\rangle$, $\phi(x) = \langle x|\phi\rangle$:

i) $\langle\phi|\psi\rangle$.
ii) $\langle\phi|A(\hat{x})|\psi\rangle$, where $A(\hat{x})$ is any function of the position operator.
iii) $\langle\phi|\hat{p}|\psi\rangle$, where $\hat{p}$ is the momentum operator.
iii) $\langle\phi|\hat{p}^2|\psi\rangle$, where $\hat{p}^2$ is any function of the momentum operator.

Consider a spinless particle moving in three-dimensions. The eigenbasis of the position operator $|\mathbf{r}\rangle = |r, \theta, \phi\rangle$ is indexed by the values of the position $\mathbf{r}$. Write the following expressions in terms of the position space wave functions $\psi(\mathbf{r}) = \langle\mathbf{r}|\psi\rangle$, $\phi(\mathbf{r}) = \langle\mathbf{r}|\phi\rangle$:

i) $\langle\phi|\psi\rangle$.
ii) $\langle\phi|A(\hat{\mathbf{r}})|\psi\rangle$, where $A(\hat{\mathbf{r}})$ is any function of the position operator.

Consider now a spinl 1/2 particle moving in three-dimensions. The eigenbasis of both the position operator and the z-component of spin $|\mathbf{r}\rangle = |r, \theta, \phi, m_s\rangle = |r, \theta, \phi\rangle \otimes |m_s\rangle$ is indexed by the values of the position $\mathbf{r}$ and $m_s = \pm 1/2$. Write the following expressions in terms of the position space wave functions $\psi_{m_s}(\mathbf{r}) = \langle\mathbf{r}, m_s|\psi\rangle$ (and similarly for $|\phi\rangle$):
i) $\langle\phi|\psi\rangle$.
ii) $\langle\phi|A(\hat{\mathbf{r}})|\psi\rangle$, where $A(\hat{\mathbf{r}})$ is any function of the position operator.
iii) $\langle\phi|A(\hat{\mathbf{r}})\hat{S}_z|\psi\rangle$, where $A(\hat{\mathbf{r}})$ is any function of the position operator.