A. Born rule for momentum probabilities

A spinless particle moves in one dimension and, at some instant, is described by the wave function \( \psi(x) = \langle x|\psi \rangle \). At that instant the momentum of the particle is measured. What are the possible outcomes of this measurement and with which probabilities (probability densities, to be more precise)?

B. Spin singlet

Consider two spin 1/2 particles. Show that the state \(|\text{singlet}\rangle = (|↑\downarrow⟩ - |↓↑⟩)/\sqrt{2} \) is an eigenstate of both \( \hat{S}^2 \) and \( \hat{S}_z \) (\( \hat{S} \) is the total spin operator). What are the corresponding eigenvalues?

C. Warming up to the Bell inequalities

One of the deepest results obtained by the Human Race is the “Bell inequality” which I hope to discuss at some point at the end of the semester. The proof hinges on a simple fact about spin-1/2 particles. Suppose two spin-1/2 particles are known to be in a spin singlet state \(|\text{singlet}\rangle\). Let \( \hat{S}_a^{(1)} = \hat{S}_a \) be the component of the spin of particle one in the direction of the unit vector \( a \) and similarly for \( \hat{S}_b^{(2)} \). Then,

\[
\langle \text{singlet}|\hat{S}_a^{(1)} \hat{S}_b^{(2)}|\text{singlet}\rangle = -\frac{\hbar^2}{4} \cos \theta,
\]

where \( \theta \) is the angle between \( a \) and \( b \). Prove the relation above.

D. More bra-ketology

Consider a spinless particle moving in one dimension. Write the expressions below in the eigenbasis of position (that is, in terms of \( \psi(x) = \langle x|\psi \rangle \)):

i) \( V(\hat{x})|\psi \rangle \), where \( V(x) \) is an analytic function of \( x \).
ii) \( \hat{p}|\psi \rangle \)
iii) \( \left[ \frac{\hat{x}^2}{2m} + V(\hat{x}) \right]|\psi \rangle = E|\psi \rangle \)

Write the expressions below in the eigenbasis of momentum (that is, in terms of \( \tilde{\psi}(p) = \langle p|\psi \rangle \)):

i) \( \hat{\tilde{x}}^2|\psi \rangle \), where \( V(x) \) is an analytic function of \( x \).
ii) \( \hat{\tilde{p}}|\psi \rangle \)
iii) \( \left[ \frac{\hat{\tilde{p}}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \right]|\psi \rangle = E|\psi \rangle \)